ANNALS OF MATHEMATICS STUDIES

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FOURIER TRANSFORMS

BY

S. BOCHNER AND K. CHANDRASEKHARAN

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PREFACE

This is a tract dealing with Fourier transforms and some topics naturally connected with them, and although the material included is familiar, if not classical, there is not much of a duplication with other books in the field.

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Princeton University

and

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CHAPTER I

FOURIER TRANSFORMS IN L_1 (ONE VARIABLE)

§1. Elementary properties

The Fourier transform of f(x) is by definition the functional

$$\phi_f(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)dx ,$$

α being a real number. The simplest class of functions f(x) for which it can be introduced is the Lebesgue class L_1 on $(-\infty,\infty)$.

If $f(x) \in L_1(-\infty,\infty)$, then $\phi_f(\alpha)$ or, briefly, $\phi(\alpha)$ exists for every α. We shall recount a few properties of $\phi(\alpha)$.

(1.1) $\phi(\alpha)$ is bounded, since

$$|\phi(\alpha)| \leq \| f \| = \int_{-\infty}^{\infty} |f(x)|dx.$$

(1.2) $\phi(\alpha)$ is uniformly continuous in $-\infty < \alpha < \infty$. If $y > 0$, then we have:

$$|\phi(\alpha+y)-\phi(\alpha)| = \left|\int_{-\infty}^{\infty} f(x)e^{i\alpha x}(e^{iyx}-1)dx\right|$$

$$\leq \int_{-\infty}^{\infty} |f(x)| \cdot |e^{iyx}-1|dx$$

$$\leq \int_{-\infty}^{\infty} |f(x)| \cdot 2\left|\sin \frac{xy}{2}\right|dx$$

$$\leq 2\left[\int_{-\infty}^{-R} + \int_{R}^{\infty}\right]|f(x)|dx + yR\int_{-R}^{R}|f(x)|dx$$

Given $\varepsilon > 0$, we can choose R so large and afterwards y so small that the last expressions add up to less than ε.

(1.3) If c_1 and c_2 are real numbers, and T is the operation which carries f into ϕ, then

$$T(c_1 f_1 + c_2 f_2) = c_1 . Tf_1 + c_2 . Tf_2$$

(1.4) $$T[f(Rx)] = \frac{1}{R}\phi\left(\frac{\alpha}{R}\right); \quad T[\bar{f}(x)] = \overline{\phi(-\alpha)},$$

where $(\overline{\quad})$ denotes the complex conjugate.

(1.5) If a sequence of functions $\{f_n(x)\} \to f(x)$ in L_1-norm, then the sequence of their Fourier transforms $\{\phi_n(\alpha)\} \to \phi(\alpha)$ uniformly in $-\infty < \alpha < \infty$.

(1.6) If $Tf_1 = \phi_1$, $Tf_2 = \phi_2$, then

$$\int_{-\infty}^{\infty} \phi_1(y)f_2(y)dy = \int_{-\infty}^{\infty} \phi_2(y)f_1(y)dy.$$

In fact

$$\int_{-\infty}^{\infty} \phi_1(y)f_2(y)dy = \int_{-\infty}^{\infty} f_2(y)\left(\int_{-\infty}^{\infty} e^{iyx}f_1(x)dx\right)dy$$

and by Fubini's theorem this is equal to

(1.61) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iyx} f_1(x) f_2(y) dx\, dy$$

since

(1.62) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_1(x)| \cdot |f_2(y)| dx\, dy$$

$$= \int_{-\infty}^{\infty} |f_1(x)| dx \cdot \int_{-\infty}^{\infty} |f_2(y)| dy < \infty .$$

However, the double integral (1.61) is symmetric in $f_1(x), f_2(x)$, which proves our assertion.

Remarks: It should be noted that conceptually the composition-theorem (1.6) is radically different from the more important convolution-theorem as embodied in theorem 2 to be proved later. In theorem 2, the functions $f_1(x)$ and $f_2(x)$ are composed by themselves and their transforms by themselves. However (1.6) composes a function with a transform, and this can be done only because, in the case of Fourier transforms, both the functions and their transforms are defined over a common space, namely the line $-\infty < y < \infty$.

§2. Riemann-Lebesgue lemma

THEOREM 1: If $f(x) \in L_1(-\infty, \infty)$ then

$$\lim_{|\alpha| \to \infty} \phi(\alpha) = 0 .$$

This theorem is usually known as the Riemann-Lebesgue lemma.

Proof: If we introduce any interval

$$I : a \leq x < b$$

and the function

$$w_I(x) = \begin{cases} 1, & \text{if } x \in I, \\ 0, & \text{if } x \notin I, \end{cases}$$

then for $f(x) = w_I(x)$, we have

$$\phi_f(\alpha) = \int_a^b e^{i\alpha x}\,dx = \frac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}$$

so that

$$|\phi_f(\alpha)| \leq \frac{2}{|\alpha|} .$$

This result holds for every step-function $f(x)$ which is constant on a finite number of (bounded) intervals and vanishes outside, on account of (1.3). These step-functions are dense in the space $L_1(-\infty,\infty)$, that is, corresponding to each $\varepsilon > 0$, there exists a step function f_ε such that

$$\| f - f_\varepsilon \| < \varepsilon .$$

Now,

$$\phi_f(\alpha) = \phi_{f-f_\varepsilon}(\alpha) + \phi_{f_\varepsilon}(\alpha), \qquad \text{(see(1.3))}$$

and

$$|\phi_f(\alpha)| \leq |\phi_{f-f_\varepsilon}(\alpha)| + |\phi_{f_\varepsilon}(\alpha)|$$
$$\leq \varepsilon + |\phi_{f_\varepsilon}(\alpha)| .$$

Thus

$$\overline{\lim_{|\alpha|\to\infty}} |\phi_f(\alpha)| \leq \varepsilon + \overline{\lim_{|\alpha|\to\infty}} |\phi_{f_\varepsilon}(\alpha)|.$$

But the second term on the right is zero for a step function f_ε and therefore

$$\overline{\lim_{|\alpha|\to\infty}} |\phi_f(\alpha)| \leq \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain the result that

$$\lim_{|\alpha|\to\infty} \phi_f(\alpha) = 0 .$$

§3. Convolution of two functions

Let $f(x), g(x) \in L_1(-\infty,\infty)$, and let $\phi(\alpha)$, $\psi(\alpha)$ be their respective Fourier transforms. The (resultant, or) convolution of f and g is defined to be

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

$$= \int_{-\infty}^{\infty} f(y)g(x-y)dy ,$$

where the second integral arises from the first if, for a fixed x, we replace the variable y by x-y .

We shall now prove a result which states that the Fourier transform of the convolution of two functions is the product of their transforms.

THEOREM 2: *If* $f, g \in L_1(-\infty,\infty)$, *then the integral defining* $h(x)$ *exists for almost all* x, *belongs to* $L_1(-\infty,\infty)$ *and*

$$\| h(x) \| \leq \| f \| . \| g \|$$

(the notation is as in (1.1)). *Furthermore, if* $\chi(\alpha)$ *denotes the Fourier transform of* $h(x)$ *then* $\chi(\alpha) = \phi(\alpha) . \psi(\alpha)$.

Proof: First we note that if $f(x)$ is measurable in x, then $f(x-y)$ is measurable in (x,y). To show that

$$h(x) = \int_{-\infty}^{\infty} f(x-t) g(t) dt$$

exist almost everywhere, we observe that

$$\int_{-\infty}^{\infty} |f(x-t)| . |g(t)| dx = |g(t)| \int_{-\infty}^{\infty} |f(y)| dy$$

$$= |g(t)| . \| f \| \in L_1(-\infty,\infty),$$

and hence

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} |f(x-t)| |g(t)| dx$$

exists, and by Fubini's theorem, it follows that

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x-t)| \, |g(t)| dt$$

exists and $h(x)$ exists almost everywhere and is of class $L_1(-\infty,\infty)$.

Furthermore,

$$\chi(\alpha) = \int_{-\infty}^{\infty} h(x) e^{i\alpha x} dx = \int_{-\infty}^{\infty} dx \; . \; e^{i\alpha x} \int_{-\infty}^{\infty} f(x-y) g(y) dy$$

$$=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}f(x-y)e^{i\alpha(x-y)}g(y)e^{i\alpha y}dy$$

$$=\int_{-\infty}^{\infty}dy.g(y)e^{i\alpha y}\int_{-\infty}^{\infty}f(x)e^{i\alpha x}dx$$

$$=\phi(\alpha)\ .\ \psi(\alpha),$$

and the justification follows again from Fubini's theorem with $f(x-y)e^{i\alpha(x-y)}g(y)e^{i\alpha y}$ in place of $f(x-y)g(y)$.

§4. Derivative of a function and its transform

One of our objects will be to prove that if

$$\phi(\alpha)=\int_{-\infty}^{\infty}f(x)e^{i\alpha x}dx$$

then

$$f(x)\sim\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ix\alpha}\ \phi(\alpha)d\alpha$$

in <u>some</u> sense. But before proving it, we will note certain heuristic consequences of the inverse relation, and establish some of them without the aid of the inverse relation. We have already noticed that

(A): if $f(x)$ has the Fourier transform $\phi(\alpha)$ then $f(x)e^{ixh}$ has the transform $\phi(\alpha+h)$, and

(B): if $f(x)$ has the Fourier transform $\phi(\alpha)$ then $f(x+h)$ has the transform $\phi(\alpha)e^{-i\alpha h}$.

Now, propositions (A) and (B) exhibit certain invertive properties, from which we deduce:

(C): $T[f(x)\ .\ \frac{e^{ixh}-1}{h}\ ]=\frac{\phi(\alpha+h)-\phi(\alpha)}{h}$,

and

$$(D): \quad T\left[\frac{f(x+h)-f(x)}{h}\right] = \phi(\alpha) \cdot \frac{e^{-i\alpha h}-1}{h} \quad .$$

Tentatively, if we let $h \to 0$ in (C) and (D) we obtain formally

$$(E): \quad T[f(x).ix] = \phi'(\alpha),$$

$$(F): \quad T[f'(x)] = -i\alpha\phi(\alpha),$$

and we shall now establish (E) and (F) on a rigorous basis.

THEOREM 3. (i) *If* $f(x) \in L_1$ *and* $ixf(x) \in L_1$, *then* $\phi'(\alpha)$ *exists and* $\phi_{ixf}(\alpha) = \phi'(\alpha)$.

(ii) *If* $f(x) \in L_1$ *and* $f'(x) \in L_1$, *then* $\phi_{f'}(\alpha) = -i\alpha\phi(\alpha)$; also

$$(4.1) \qquad f(x) = -\int_x^\infty f'(x)dx$$

Proof: (i) We have

$$\frac{\phi(\alpha+h) - \phi(\alpha)}{h} = T\left[f(x) \cdot \frac{e^{ixh}-1}{h}\right]$$

$$= T[f_h(x)], \text{ say.}$$

Now, $f_h(x) \to ixf(x)$ in L_1-norm, because $f_h(x) \to ixf(x)$ at every point x, and

$$|f_h(x)| \leq |f(x)| \left|\frac{e^{ixh}-1}{h}\right| \leq |x| \cdot |f(x)| \in L_1 \quad .$$

On applying property (1.5) we get

$$T[f_h(x)] \longrightarrow T[ixf(x)],$$

uniformly, as $h \to 0$. Hence, at every point α, there exists the derivative $\phi'(\alpha)$ in the ordinary sense, and

$$\phi_{ixf}(\alpha) = \phi'(\alpha) .$$

(ii) The precise meaning of our assumptions is that there exists a function $g(x) \in L_1$ which we choose to denote by $f'(x)$ and an indefinite integral of it

$$f(x) = \int^{x} g(y)dy$$

such that

$$f(x) \in L_1(-\infty,\infty).$$

Now,

$$f(A) - f(a) = \int_{a}^{A} g(x)dx$$

If we keep a fixed, and let $A \to \infty$, since $g(x) \in L_1$, we have

$$\int_{a}^{A} g(x)dx \longrightarrow c.$$

Therefore, $f(A) \to l$; similarly $f(-A) \to -m$, say. Since $f(x) \in L_1(-\infty,\infty)$ we must have $l = -m = 0$, and this, first of all, proves (4.1).

Now, if $T[f'(x)] = \psi(\alpha)$, then

$$\psi(\alpha) = \lim_{A\to\infty} \int_{-A}^{A} e^{i\alpha x} df(x)$$

$$= \lim_{A\to\infty} [\{e^{i\alpha x} f(x)\}_{-A}^{A} - i\alpha \int_{-A}^{A} e^{i\alpha x} f(x)dx].$$

But the boundary terms vanish because, $l = -m = 0$, and thus

$$\psi(\alpha) = -i\alpha\phi_f(\alpha)$$

as claimed.

Remark: There is a much stronger theorem stating that if $f(x) \in L_1$ and $f^{(r)}(x) \in L_1$, then $f^{(1)}(x), \ldots, f^{(r-1)}(x) \in L_1$. We shall postpone this to a later context.

§5. Inversion formula

We wish to give some simple conditions under which

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \phi(\alpha) d\alpha$$

at a given point x, assuming that $f(x) \in L_1(-\infty, \infty)$.

Let

(5.1)
$$S_R(x) = \frac{1}{2\pi} \int_{-R}^{R} e^{-i\alpha x} \phi(\alpha) d\alpha$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t} \cdot f(x+t) dt$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin Rt}{t} \cdot \frac{f(x+t) + f(x-t)}{2} dt .$$

Let

(5.2)
$$g_x(t) = \frac{f(x+t) + f(x-t)}{2} - f(x).$$

Then

(5.3)
$$S_R(x) - f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin Rt}{t} g_x(t) dt.$$

THEOREM 4: If

$$\int_0^{\delta} \left| \frac{g_x(t)}{t} \right| dt < \infty ,$$

then

$$\lim_{R \to \infty} S_R(x) = f(x) .$$

Proof: Let $\delta > 0$ be fixed. Then,

$$S_R(x) - f(x) = \frac{2}{\pi}\left[\int_0^\delta + \int_\delta^\infty\right] \frac{g_x(t)}{t} \sin Rt\, dt$$

$$= I_1 + I_2 \text{ , say.}$$

Now, $I_2 = o(1)$ by the Riemann-Lebesgue lemma. And, since $\frac{|g_x(t)|^2}{t}$ is absolutely integrable in $(0,\delta)$, it follows that $I_1 = \varepsilon(\delta) \to 0$, as $\delta \to 0$.

Remarks: Theorem 4 shows that if $f(x) \in L_1$, then the convergence of $S_R(x)$ to $f(x)$ at a point depends only on the behaviour of $f(x)$ in a neighborhood of that point. This is Riemann's localization theorem.

Note that the condition of the theorem is fulfilled, if $f(x)$ has right and left derivatives at the point in question, provided that $f(x)$ is normalized as

$$\tfrac{1}{2}[f(x+0) + f(x-0)].$$

§6. Uniqueness of Fourier transform

THEOREM 5: *If* $f(x) \in L_1(-\infty,\infty)$ *and* $\phi(\alpha) = 0$ *for every* α, *then* $f(x) = 0$ *almost everywhere*.

Proof: Let $g_{a,\varepsilon}(x)$ be the function defined as follows:

$$g_{a,\varepsilon}(x) = \begin{cases} 1, & -a \le x \le a ; \\ 0, & x > a+\varepsilon,\ x < -a-\varepsilon; \\ \frac{a-x+\varepsilon}{\varepsilon}, & a \le x \le a+\varepsilon; \\ \frac{x+a+\varepsilon}{\varepsilon}, & -a-\varepsilon \le x \le -a . \end{cases}$$

Let

$$T[g_{a,\epsilon}(x)] = \psi_{a,\epsilon}(\alpha),$$

so that

$$\psi_{a,\epsilon}(\alpha) = 2\int_0^\infty g_{a,\epsilon}(x) \cos x\alpha \, dx$$
$$= -\frac{2}{\alpha}\int_0^\infty g'_{a,\epsilon}(x) \sin x\alpha \, dx$$
$$= \frac{2}{\alpha\epsilon} \int_a^{a+\epsilon} \sin x\alpha \, dx.$$

Hence

$$\psi_{a,\epsilon}(\alpha) = 0\left(\frac{1}{\alpha^2}\right), \text{ as } |\alpha| \to \infty.$$

Therefore, since $\psi_{a,\epsilon}(\alpha)$ is (continuous and) bounded, we have

(6.1) $$\psi_{a,\epsilon}(\alpha) \in L_1(-\infty,\infty).$$

Also, $g_{a,\epsilon}(x)$ satisfies the condition of theorem 4 at every point $-\infty < x < \infty$. Therefore by that theorem we have

(6.2) $$g_{a,\epsilon}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\alpha} \psi_{a,\epsilon}(\alpha) d\alpha$$

Furthermore, on account of (6.1), the integral on the right of (6.2) converges absolutely. Finally,

(6.3) $$\int_{-\infty}^{\infty} f(y) g_{a,\epsilon}(x-y) dy = \int_{-\infty}^{\infty} \phi(\alpha)\psi_{a,\epsilon}(\alpha) e^{-i\alpha x} d\alpha .$$

This follows (either by the composition-rule in (1.6) or directly) if we substitute for $g_{a,\epsilon}$ the integral in (6.2) and interchange the integration, which is justified by the absolute convergence of (6.2).

Since $\phi(\alpha) = 0$, we have:

$$\int_{-\infty}^{\infty} f(y) g_{a,\varepsilon}(x-y)dy = 0,$$

or,

$$\int_{x-a}^{x+a} f(y)dy = 0, \text{ for each } a,$$

or,

$$\int_{\alpha}^{\beta} f(y)dy = 0, \text{ for all } \alpha \text{ and } \beta.$$

Hence $f(x) = 0$ for almost all x; that is, $f(x)$ is the null-function of the class $L_1(-\infty,\infty)$ or, rather, the null-element of the Banach space $L_1(-\infty,\infty)$.

Remarks: We will easily extend this theorem to several variables, as will be seen in the next chapter.

In one variable, however, there is a deeper theorem in which the (absolute) integrability of $f(x)$ in $(-\infty,\infty)$ is entirely dispensed with and only the existence of the Cauchy limit

$$\phi(\alpha) = \lim_{A\to\infty} \int_{-A}^{A} e^{i\alpha x} f(x)dx$$

for every α is presupposed. This type of theorem has, apparently, never been extended to more than one variable, and any non-trivial result in this direction would be very desirable indeed.

§7. Summability theorems

Let $f(x) \in L_1$, $K(\alpha) \in L_1$. We know that if $T[f(x)] = \phi(\alpha)$, then $T[f(x+t)] = \phi(\alpha)e^{-it\alpha}$. Let $T[K(\alpha)] = H(t)$.

Then $T[K(\frac{\alpha}{R})] = R\, H(Rt)$. Assume:

(7.1) $$K(\alpha) \in L_1(-\infty,\infty)$$

(7.2) $K(0) = 1$, $K(\alpha)$ is continuous at $\alpha = 0$

(7.3) $K(\frac{\alpha}{R})$ can be "inverted" at the origin; that is

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} R\, H(Rt)dt$$

(7.4) $K(\alpha)$ is even.

Define:

(7.5) $$S_R^K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\alpha)e^{-ix\alpha}K(\tfrac{\alpha}{R})d\alpha .$$

We then have the following

<u>LEMMA 1</u>:

(7.6) $$S_R^K(x) - f(x) = \frac{1}{\pi}\int_0^{\infty} g_x(t)RH(Rt)dt,$$

where $g_x(t)$ is defined as in (5.2).

Proof:

$$S_R^K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+t)RH(Rt)dt$$

on account of property (1.6). If we now use assumption (7.3) on $K(\alpha)$, we obtain:

$$S_R^K(x) - f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{f(x+t) - f(x)\}RH(Rt)dt.$$

By assumption (7.4), H(t) is even; so that

$$S_R^K(x) - f(x) = \frac{1}{\pi}\int_0^{\infty} g_x(t)RH(Rt)dt.$$

LEMMA 2: If $H(t) \geq 0$, then $S_R^K(x) \in L_1(-\infty, \infty)$ as a function in x (even for small R).

For,

$$\| S_R^K(x) \| \leq \frac{\| f(x) \|}{2\pi} \int_{-\infty}^{\infty} RH(Rt)dt$$

$$= \| f \| .$$

Let us now specialize the function $K(\alpha)$ and see how (7.6) will read.

(7.7) $$K(\alpha) = \begin{cases} 1, & -1 \leq \alpha \leq 1 \\ 0, & \text{outside} \end{cases}$$

$$H(t) = 2 \int_0^1 \cos \alpha t \, d\alpha = \frac{2 \sin t}{t}$$

$$S_R^K(x) = \frac{1}{2\pi} \int_{-R}^{R} e^{-i\alpha x} \phi(\alpha) d\alpha$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{\sin Rt}{t} dt$$

$$= S_R(x)$$

in the notation of §5.

(7.8) $$K(\alpha) = \begin{cases} 1 - |\alpha|, & |\alpha| \leq 1, \\ 0, & |\alpha| > 1. \end{cases}$$

$$H(t) = 2 \int_0^1 (1-\alpha) \cos \alpha t \, d\alpha = 2 \int_0^1 (1-\alpha) d_\alpha \frac{\sin \alpha t}{t}$$

$$= \frac{2}{t} \int_0^1 \sin \alpha t \, d\alpha = \frac{2}{t} \frac{1 - \cos t}{t} = \left(\frac{\sin t/2}{t/2} \right)^2$$

$$S_R^K(x) = \frac{1}{2\pi}\int_{-R}^{R}(1-\frac{|\alpha|}{R})\, e^{-i\alpha x}\,\phi(\alpha)d\alpha$$

$$= \int_0^R (1-\frac{\alpha}{R})\,(\frac{\phi(\alpha)e^{-i\alpha x} + \phi(-\alpha)e^{i\alpha x}}{2})d\alpha$$

$$= \int_0^R (1-\frac{\alpha}{R})d_\alpha\, S_\alpha(x)$$

$$= \frac{1}{R}\int_0^R S_\alpha\, d\alpha\ .$$

(7.9) $K(\alpha) = e^{-|\alpha|}$, $H(t) = 2\int_0^\infty e^{-\alpha}\cos\alpha t\ dt = \frac{2}{1+t^2}$

(7.10) $K(\alpha) = e^{-\alpha^2}$, $H(t) = \int_{-\infty}^{\infty} e^{-\alpha^2+i\alpha t}d\alpha = e^{-t^2/4}\,\pi^{1/2}$.

<u>THEOREM 6</u>: <u>If</u> (1) $K(\alpha) \in L_1(-\infty,\infty)$, (2) $K(0) = 1$, (3) $K(\alpha)$ <u>is continuous at</u> $\alpha = 0$ <u>and</u> (4) $K(\alpha) = K(-\alpha)$; <u>if</u> (5) $H(t)$ <u>is monotonely decreasing in</u> $0 \leq t < \infty$ <u>and</u> (6) $\int_0^\infty H(t)dt < \infty$, <u>and hence</u> (7) $tH(t) \to 0$ <u>as</u> $t \to \infty$, <u>and if furthermore</u> (8) $1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(t)dt$; <u>then, at a point</u> x, <u>the condition</u>

$$\frac{1}{h}\int_0^h g_x(t)dt = o(1), \text{ as } h \to 0$$

<u>implies</u>

$$S_R^K(x) - f(x) = o(1), \text{ as } R \to \infty .$$

Note that H(t) is even, since $K(\alpha)$ is even, and that, as will be proved subsequently in theorem 9, assumption (8) is a consequence of the previous ones.

Proof: From (7.6) we have

$$S_R^K(x) - f(x) = \frac{1}{\pi}\int_0^\infty g_x(t)RH(Rt)dt$$

$$= \int_0^u + \int_u^\infty$$

$$= I_1 + I_2, \text{say}.$$

If we put

$$G(t) = \int_0^t g_x(s)ds,$$

we have

$$I_1 = \int_0^u RH(Rt)dG$$

$$= uR \,.\, H(Ru)\frac{G(u)}{u} - \int_0^u G(t)R\, d_t\, H(Rt)$$

$$= o(1) + O(\int_0^u tR\, d_t H(Rt))$$

$$= o(1) + O(uR \,.\, H(Ru) - \int_0^u RH(Rt)dt)$$

and finally

(7.11) $$I_1 = o(1),$$

by assumptions (6) and (7). Also

$$I_2 = \frac{1}{\pi}\int_u^\infty [\frac{f(x+t)+f(x-t)}{2} - 2f(x)]RH(Rt)dt$$

$$= I_{2,1} + I_{2,2} + I_{2,3}, \text{ say}.$$

$$|I_{2,1}| \leq \frac{1}{2\pi} RH(Ru)\int_u^\infty |f(x+t)|dt$$

$$\leq \frac{1}{2\pi} \,.\, uR \,.\, H(Ru) \,.\, \frac{1}{u}\int_{-\infty}^\infty |f(x+t)|dt$$

(7.12) $< \varepsilon$, as $R \to \infty$, for fixed u.

Similarly,

(7.13) $|I_{2,2}| < \varepsilon$.

$$|I_{2,3}| \leq \frac{R|f(x)|}{\pi} \int_u^\infty H(Rt)dt = \frac{|f(x)|}{\pi} \int_{uR}^\infty H(t)dt = o(1),$$

as $R \to \infty$, because of assumption (7).

The result now follows from (7.11) to (7.13)

THEOREM 7. Assumptions: (1) $K(\alpha)$ is the same as in theorem 6, and $H(t)$ satisfies assumption (8); (2) there exists a function $H_o(t)$ satisfying assumptions (5),(6) and (7) of $H(t)$ in theorem 6 such that $|H(t)| \leq H_o(t)$.

Conclusion: if

$$\frac{1}{h} \int_0^h |g_x(t)|dt = o(1), \quad \text{as } h \to 0,$$

then

$$S_R^K(x) - f(x) = o(1), \quad \text{as } R \to \infty.$$

The proof is similar to that of theorem 6; we have only to use the fact:

$$|g_x(t)| \leq \left|\frac{f(x+t)}{2}\right| + \left|\frac{f(x-t)}{2}\right| + |f(x)|.$$

Remarks: In theorem 7 the assumption on $K(\alpha)$ and assumption (8) were needed only to secure the integral representation of $S_R^K(x)$.

Theorem 6 covers the cases when $K(\alpha) = e^{-|\alpha|}$ (Abel) and $K(\alpha) = e^{-\alpha^2}$ (Gauss) but <u>not</u> (Fejer):

$$K(\alpha) = \begin{cases} 1 - |\alpha|, & |\alpha| \leq 1, \\ 0, & |\alpha| > 1. \end{cases}$$

Theorem 7 covers the last case, because

$$\left(\frac{\sin t/2}{t/2} \right)^2 \leq \frac{c}{1+t^2} .$$

Since $\frac{1}{h} \int_0^h g_x(t)dt \to 0$ if and only if $\frac{1}{2h} \int_{-h}^h f(x+t)dt \to f(x)$, and the latter holds almost everywhere for $f(x) \in L_1$ by the fundamental theorem on absolute continuity of the indefinite integral, therefore theorem 6 proves that the Fourier transform of an L_1-function is Abel- (or Gauss -)summable almost everywhere.

Even the more stringent condition $\frac{1}{h} \int_0^h |g_x(t)|dt \to 0$ is satisfied almost everywhere by a further theorem of Lebesgue, and hence by theorem 7, the Fourier transform of an L_1-function is summable (C,1) almost everywhere.

§8. <u>Some applications of summability theorems</u>.

<u>THEOREM</u> 8. <u>If</u> $f(x) \in L_1$ <u>and</u> $\phi(\alpha) \in L_1$, <u>then</u>

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\alpha} \phi(\alpha)d\alpha = f(x),$$

<u>almost everywhere</u>.

Proof: By theorem 6, we have, for almost all x,

$$(8.1) \qquad f(x) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\alpha} e^{-\frac{|\alpha|}{R}} \phi(\alpha)d\alpha .$$

Since $\phi(\alpha) \in L_1$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \sup_R |e^{-ix\alpha}e^{-\frac{|\alpha|}{R}}\phi(\alpha)|d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}|e^{-ix\alpha}\phi(\alpha)|d\alpha < \infty.$$

Hence in (8.1) we may let $R \to \infty$ under the integral-sign and it follows by "majorized limits" that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\alpha}\phi(\alpha)d\alpha$$

almost everywhere.

Remark: If we define for all x,

$$f_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\alpha}\phi(\alpha)d\alpha$$

then, by the uniqueness theorem (5), $f(x) = f_0(x)$ almost everywhere, and thus it follows that $f(x)$ differs from a continuous function on a set of measure zero.

THEOREM 9: *If $f(x) \in L_1$ and $|f(x)| \leq M$ in $-h \leq x \leq h$, $h > 0$, and $\phi(\alpha) \geq 0$, then*

$$\int_{-\infty}^{\infty}|\phi(\alpha)|d\alpha = \int_{-\infty}^{\infty}\phi(\alpha)d\alpha < \infty .$$

Proof: We assume first that

$$|f(x)| < N, \quad \text{in } (-\infty,\infty),$$

and put $K(\alpha) = e^{-|\alpha|}$. Then

$$|S_R^K(x)| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+\frac{t}{R})H(t)dt\right|$$

$$< \frac{N}{2\pi}\int_{-\infty}^{\infty} H(t)dt = N$$

so that

$$|S_R^K(o)| < N ;$$

or,

(8.2) $$\int_{-\infty}^{\infty} \phi(\alpha) e^{-\frac{|\alpha|}{R}} d\alpha < 2\pi N$$

Due to $\phi(\alpha) \geq 0$, we may pass to the limit as $R \to \infty$ under the integral sign, and obtain

(8.3) $$\int_{-\infty}^{\infty} \phi(\alpha)\, d\alpha < 2\pi N$$

which proves the assertion.

Actually it is enough to assume that $|f(x)| < M$ in $(-h, h)$, for

$$S_R^K(x) = \int_{-hR}^{hR} + \int_{-\infty}^{-hR} + \int_{hR}^{\infty} ;$$

at $x = 0$, we obtain

$$\left| \int_{-hR}^{hR} \right| \leq M$$

as before. Then by using the property $|H(t)| < \frac{A}{t}$ for $h < t < \infty$ we obtain an estimate

$$\left| \int_{-\infty}^{-hR} + \int_{hR}^{\infty} \right| \leq M_2$$

uniformly for $1 \leq R < \infty$, and we obtain the conclusions (8.2) and then (8.3) with $M_1 + M_2$ instead of N.

Remarks: From theorems 8 and 9 we observe the following:

(A): If $f(x) \in L_1$ and $f(x)$ is continuous, and $\phi(\alpha) \in L_1$, then inversion holds everywhere.

(B): If $f(x) \in L_1$, is bounded and continuous and has a positive (that is, non-negative) transform, then inversion holds <u>everywhere</u>.

§9. Continuity in norm

Let

$$f(x) \in L_1, \quad f_h(x) = f(x+h), \quad \| f \| = \int_{-\infty}^{\infty} |f(x)| dx = A;$$

then, $\| f_h \| = \| f \|$. Let

$$w(h) = \| f(x) - f_h(x) \|$$

$$= \int_{-\infty}^{\infty} |f(x) - f(x+h)| dx \quad .$$

Then, $w(t)$ is even and bounded; $0 \leq w(h) \leq 2A$; $w(h_1 + h_2) \leq w(h_1) + w(h_2)$; $w(0) = 0$.

<u>THEOREM 10</u>. $w(h) \to 0$ as $h \to 0$.

Proof: Let $f(x) = 1$ in (a,b) and $f(x) = 0$ outside (a,b). Then

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)| dx = \int_a^{a+h} dx + \int_b^{b+h} dx = 2h \to 0, \text{as } h \to 0.$$

Since

$$w_{f_1+f_2}(h) \leq w_{f_1}(h) + w_{f_2}(h)$$

the result is true for step-functions. Again let $\{f_n\}$ be a sequence of step-functions such that $f_n \to f$; then

$$w_f(h) \leq w_{f-f_n}(h) + w_{f_n}(h)$$

$$\leq 2\| f - f_n \| + w_{f_n}(h)$$

$$\leq \varepsilon_n + \varepsilon(h) \to 0.$$

THEOREM 11. *If f(x), g(x) are bounded functions of class L_1, then the integral*

$$h(x) = \int_{-\infty}^{\infty} f(x+y)\overline{g(y)}\, dy = \int_{-\infty}^{\infty} f(x-y)\overline{g(-y)}\, dy$$

exists for every x, and h(x) is bounded, continuous and of class $L_1(-\infty, \infty)$.

Proof:

$$\left| \int_{-\infty}^{\infty} f(x+y)\overline{g(y)}\, dy \right| \leq \int_{-\infty}^{\infty} |f(x+y)|\, dy \,.\, \sup |g(y)|$$

$$\leq M \,.\, \|f\|$$

thus h(x) is bounded; it is also continuous, for

$$|h(x) - h(x+t)| \leq \int_{-\infty}^{\infty} |f(y+x) - f(y+x+t)| . |g(y)|\, dy$$

$$\leq \sup |g(y)| . w_f(t),$$

and by the previous theorem, $w(t) \to 0$ as $t \to 0$, so that

$$|h(x) - h(x+t)| = \varepsilon(t) = o(1), \text{ as } t \to 0.$$

THEOREM 12. *If f(x) is a bounded function of class L_1, then*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(\alpha)|^2 d\alpha$$

the integrals being finite.

Proof: Setting up the function

$$h(x) = \int_{-\infty}^{\infty} f(x+y)\overline{f(y)}dy$$

we observe that by theorem 2,

$$T[h(x)] = |\phi(\alpha)|^2 \geq 0.$$

By theorem 11, h(x) is bounded and continuous and of class L_1. It has a positive transform. Hence by theorem 9, inversion holds everywhere,

$$\int_{-\infty}^{\infty} f(y+x)\overline{f(y)}dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\phi(\alpha)|^2 e^{-i\alpha x} d\alpha .$$

Setting x = 0, we get the required result.

<u>THEOREM 13</u>. <u>If</u> f(x), g(x) <u>are both bounded functions of class</u> L_1, <u>then</u>

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\alpha)\overline{\psi(\alpha)}d\alpha .$$

Proof: By theorem 12,

$$\int_{-\infty}^{\infty} |\phi(\alpha)|^2 d\alpha < \infty, \quad \int_{-\infty}^{\infty} |\psi(\alpha)|^2 d\alpha < \infty .$$

Hence by Schwarz's inequality,

$$\int_{-\infty}^{\infty} \phi(\alpha)\overline{\psi(\alpha)}d\alpha$$

exists. Hence, at <u>every</u> point (since h(x) is continuous by theorem 11) the inversion-formula holds (cf. theorem 8) for the convolution h(x), that is:

$$h(x) = \int_{-\infty}^{\infty} f(x+y)\overline{g(y)}dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\alpha)\overline{\psi(\alpha)}e^{-i\alpha x}\, d\alpha .$$

Setting x = 0, we obtain the required result.

§10. Summability in norm.

THEOREM 14. *Let* $f(x) \in L_1$; *let* $K(\alpha)$, $H(t)$ *satisfy the assumptions of theorem* 6. *Then*

$$\| S_R^K(x) - f(x) \| \to 0$$

as $R \to \infty$.

Proof: By lemma 2 of §7, $S_R^K(x) \in L_1(-\infty,\infty)$ and

$$|S_R^K(x) - f(x)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+t) - f(x)| R|H(Rt)|dt.$$

Hence

$$\int_{-\infty}^{\infty} |S_R^K(x) - f(x)|dx$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x+t) - f(x)| R|H(Rt)|dt$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} w(t)\, R|H(Rt)|dt$$

$$= \frac{1}{\pi} \int_{0}^{\infty} w(t)\, R|H(Rt)|dt$$

$$= o(1)$$

as $R \to \infty$, since $w(t)$ is bounded and $w(t) \to 0$ as $t \to 0$, while $tH(t) \to 0$ as $t \to \infty$.

Remarks: We have proved that under suitable conditions:

(i) $S_R^K(x) \to f(x)$ almost everywhere, as $R \to \infty$;

(ii) $S_R^K(x) \to f(x)$ in L_1-norm.

§11. Derivatives of a function and their transforms

We recall theorem 3, which says:

(i) if $f(x) \in L_1$ and $ixf(x) \in L_1$, then

$$\phi_{ixf} = \phi_f'(\alpha);$$

(ii) if $f(x) \in L_1$, $g(x) \in L_1$, and $g(x) = f'(x)$, then

$$\psi(\alpha) = -i\alpha\,\phi(\alpha).$$

We now wish to prove

THEOREM 15. *If* $f(x) \in L_1$, $g(x) \in L_1$ *and* $\psi(\alpha) = T[g] = -i\alpha\,\phi(\alpha)$, *then* $g(x) = f'(x)$ *almost everywhere, that is:*

$$f(x) = -\int_x^{\infty} g(y)dy.$$

Proof: Special case: Assume that $\phi(\alpha)$, $\psi(\alpha) \in L_1(-\infty, \infty)$. In this case, we can use the inversion-formula, and hence, for almost all x,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\alpha}\,\phi(\alpha)d\alpha\ ,$$

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\alpha}\cdot -i\alpha\phi(\alpha)d\alpha$$

so that

$$\int_a^b g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{-ib\alpha} - e^{-ia\alpha})\phi(\alpha)d\alpha$$

$$= f(b) - f(a)$$

hence

$$g(x) = f'(x)$$

almost everywhere.

<u>General case:</u> Let $K(\alpha) = e^{-\alpha^2}$ and put

$$f_R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\,\phi(\alpha)K(\tfrac{\alpha}{R})d\alpha$$

$$g_R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\alpha}\cdot -i\alpha\phi(\alpha)K(\tfrac{\alpha}{R})d\alpha$$

By lemma 2, §7, $f_R(x)$, $g_R(x) \in L_1(-\infty,\infty)$. Also

$$f_R(b) - f_R(a) = \int_a^b g_R(x)dx.$$

Hence

$$f_R(x) = -\int_x^{\infty} g_R(y)dy.$$

But

(11.1) $$\lim_{R\to\infty}\int_{-\infty}^{\infty}|g_R(y) - g(y)|dy = 0$$

by theorem 14. Therefore, in particular, for every fixed x, we have

$$\lim_{R\to\infty} f_R(x) = \lim_{R\to\infty} -\int_x^{\infty} g_R(y)dy = -\int_x^{\infty} g(y)dy.$$

On the other hand, $f_R(x)$ converges almost everywhere to $f(x)$ by theorem 6. Thus, for almost all x,

$$f(x) = -\int_x^{\infty} g(y)dy,$$

which implies our assertion.

<u>THEOREM 16</u>. <u>If</u> $f(x) \in L_1$, $g(x) \in L_1$, $T[f] = \phi(\alpha)$ <u>and</u> $T[g] = \psi(\alpha)$, <u>and if</u> $\psi(\alpha) = (-i\alpha)^n\phi(\alpha)$ <u>where</u> n <u>is a positive integer, then</u> f <u>has</u> n <u>derivatives all belonging to class</u> L_1.

Note that the conclusion implies that $(-i\alpha)^k\phi(\alpha)$, $k = 0,\ldots,n$ are all transforms.

Proof: We will first show that if $\phi(\alpha) \in T$ and $(-i\alpha)^n\phi(\alpha) \in T$, $n \geq 2$, then $(-i\alpha)\phi(\alpha) \in T$ [$\phi \in T$ means ϕ is a transform].

For, $\frac{1}{\alpha+i}$ is a transform and so by theorem 2,

$$\frac{1}{(\alpha+i)^{n-1}} \in T$$

and hence, again by theorem 2,

$$\frac{(-i\alpha)^n\,\phi(\alpha)}{(\alpha+i)^{n-1}} \in T .$$

But

$$\frac{(-i\alpha)^n\,\phi(\alpha)}{(\alpha+i)^{n-1}} = \left\{ -i\alpha + A + \sum_{j=0}^{n-1} \frac{A_j}{(\alpha+i)^j} \right\}\phi(\alpha).$$

The left side belongs to T, and the last two terms on the right side also belong to T; hence

$$(-i\alpha)\phi(\alpha) \in T.$$

We next apply this result successively, and observe that

$$(-i\alpha)^k\phi(\alpha) \in T, \qquad k = 0,\ldots,n.$$

Hence, by theorem 15, we could successively derive f(x) within $L_1(-\infty, \infty)$.

<u>THEOREM 17</u>. <u>If</u> $f(x) \in L_1$ <u>and has</u> n <u>derivatives and</u> $f^{(n)}(x) \in L_1$, <u>then</u> $f^{(k)}(x) \in L_1$, $0 < k < n$.

Proof: By hypothesis,

$$f^{(k)}(x) = O(|x|^n) \quad \text{as } |x| \to \infty .$$

Let $K(\alpha) = e^{-\alpha^2}$; let

$$f_R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \phi(\alpha) e^{-\alpha^2/R^2} d\alpha.$$

As in (7.6) we have

$$f_R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-t) R\, H(Rt) dt,$$

where H is the transform of K. Set R = 1, and then we have:

$$(11.2) \quad f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \phi(\alpha) e^{-\alpha^2} d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x+t) H(t) dt.$$

By successive differentiations of the first integral, we obtain

$$(11.3) \quad D^k f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} (-i\alpha)^n \phi(\alpha) e^{-\alpha^2} d\alpha$$

and as a rule such formal differentiation under the integral-sign is admissible whenever the derived integral is convergent uniformly in every finite x-interval. Since

$$f^{(k)}(x) = O(|x|^n) \quad \text{as } x \to \infty, \quad 0 < k < n$$

and $H(t) = e^{-t^2/4}$, this rule also applies to the second integral in (11.2) so that

$$D^n f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^{(n)}(x+t)H(t)dt;$$

and H(t) being even, this is:

$$D^n f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^{(n)}(x-y)H(y)dy. \tag{11.4}$$

But $f^{(n)}(x-y)$ and $H(y)$ both belong to L_1, and then, by (1.6), (11.4) implies

$$D^n f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \psi(\alpha) e^{-\alpha^2} d\alpha \tag{11.5}$$

The two functions, $(-i\alpha)^n \phi(\alpha)e^{-\alpha^2}$, $\psi(\alpha)e^{-\alpha^2}$ appearing in (11.3) and (11.5) both belong to $L_1(-\infty,\infty)$, and, on replacing i by -i, the function $D^n f_1(x)$ appears as the common Fourier transform of both. By theorem [illegible] and continuity in α, we conclude that they are equal and hence:

$$(-i\alpha)^n \phi(\alpha) = \psi(\alpha).$$

Thus,

$$T[f^{(n)}(x)] = (-i\alpha)^n \phi(\alpha).$$

Now, using theorem 16, we observe that each of the derivatives $f^{(k)}(x)$, $k = 0,\ldots,n-1$, belongs to the class $L_1(-\infty,\infty)$.

§12. Degree of approximation

We proved that if

$$\frac{1}{h} \int_0^h g_x(t)dt = o(1) \quad \text{as } h \to 0$$

then

$$S_R^K(x) - f(x) = o(1) \text{ as } R \to \infty .$$

We now assume that

$$\int_0^h g_x(t)dt = o(h^{1+q}), \quad q > 0$$

and desire to conclude that

$$R^q(S_R^K(x) - f(x)) = o(1).$$

<u>THEOREM 18</u>. <u>The assumptions on</u> $K(\alpha)$ <u>and</u> $H(t)$ <u>are the same as before</u> (§7) <u>except for</u> $\int_0^\infty H(t)t^q dt < \infty$ <u>instead of only</u> $\int_0^\infty H(t)dt < \infty$. <u>We then have</u>:

(A): $$\int_0^h g_x(t)dt = o(h^{1+q}),$$

<u>implies</u>

$$|S_R^K(x) - f(x)| = o(\frac{1}{R^q})$$

(B): <u>If</u> $|H(t)| \leq H_0(t)$ <u>where</u> $H_0(t)$ <u>satisfies the main assumptions, then</u>:

$$\int_0^h |g_x(t)|dt = o(h^{1+q})$$

<u>implies</u>

$$|S_R^K(x) - f(x)| = o(\frac{1}{R^q}) .$$

Proof: We first observe that if $q > 0$ is any constant, and $H(t)$ decreases monotonely to zero, and

$$\int_1^\infty H(t)t^q \, dt < \infty$$

then

(18.1) $$H(t)t^{q+1} \to 0 \quad \text{as } t \to \infty .$$

For,

$$\int_{x/2}^{x} H(t)t^q \, dt \geq H(x)\int_{x/2}^{x} t^q \, dt = H(x)x^{q+1} \int_{1/2}^{1} u^q \, du \to 0$$

as $x \to \infty$. To prove the theorem, consider

$$\int_0^\infty g_x(t)R^{q+1}H(Rt)dt = \int_0^u + \int_u^\infty = I_1 + I_2, \text{ say.}$$

Regarding I_2, we have:

$$\int_u^\infty |f(x+t)|R^{q+1}H(Rt)dt$$

$$\leq R^{q+1}H(Ru) \int_u^\infty |f(x+t)|dt$$

$$\leq \frac{c(uR)^{q+1} \; H(Ru)}{u^{q+1}}$$

$$= o(1),$$

as $R \to \infty$. Furthermore,

$$\int_u^\infty |f(x)|R^{q+1} \; H(Rt)dt$$

$$\leq |f(x)| \int_u^\infty R^{q+1} \; H(Rt)dt$$

$$\leq \frac{|f(x)|}{u^q} \int_u^\infty R^{q+1} \; t^q \; H(Rt)dt$$

$$= \frac{|f(x)|}{u^q} \int_{uR}^{\infty} t^q \, H(t)dt$$

$$= o(1),$$

for fixed u, as $R \to \infty$, since the last integral converges absolutely. Hence

$$I_2 = o(1).$$

Next, put

$$G(t) = \int_0^t g_x(y)dy$$

and then,

$$I_1 = \int_0^u R^{q+1} \, H(Rt)dG(t)$$

$$= \frac{u^{q+1} R^{q+1} H(Ru) G(u)}{u^{q+1}} + I_3 \, ,$$

where

$$I_3 = \int_0^u R^{q+1} G(t)dH(tR) \, .$$

Due to the monotoneity of H(t), we may majorize G(t) under the integral sign thus obtaining

$$I_3 = o(\int_0^u R^{q+1} \, t^{1+q} \, d_t \, H(tR))$$

$$= o(R^{q+1} u^{1+q} \, H(uR) + o(\int_0^u R^{q+2} t^{1+q} H(tR)dt)$$

and it follows from all our assumptions on H(t) that

$$I_1 = o(1).$$

<u>Examples</u>: Since $H(t)t^{q+1}$ must be "small at ∞", the approximation is not optimal either for

$$K(\alpha) = \begin{cases} 1 - |\alpha|, & |\alpha| < 1 \\ 0, & |\alpha| \geq 1, \end{cases}$$

in which case

$$H(t) = \left(\frac{\sin t/2}{t/2}\right)^2 ,$$

or for $K(\alpha) = e^{-|\alpha|}$ in which case

$$H(t) = \frac{1}{1+t^2} ;$$

but it is a very good one for $K(\alpha) = e^{-\alpha^2}$, since

$$H(t) = e^{-t^2/4} ,$$

and also for $K(\alpha) = \frac{1}{1+\alpha^2}$ since $H(t) = e^{-|t|}$.

§13. <u>Abel's theorem</u>

Let

$$\frac{1}{2\pi}\left[e^{-ix\alpha}\phi(\alpha) + e^{ix\alpha}\phi(-\alpha)\right] = a(\alpha)$$

$$A_o(y) = \int_o^y a(\alpha)d\alpha$$

then for given x,

$$S_R^K \equiv S_R^K(x) = \int_o^\infty a(\alpha)K(\tfrac{\alpha}{R})d\alpha = \int_o^\infty K(\tfrac{\alpha}{R})d\,A_o(\alpha) .$$

THEOREM 19. Assumptions: (1) Let $K(\alpha)$ be continuous and differentiable in $0 \leq \alpha < \infty$, (2) $K(0) = 1$, (3) $K(y) \to 0$ as $y \to \infty$, and (4) $\int_0^\infty |K'(\alpha)|\,d\alpha < \infty$.

Conclusion: if $A_0(y) \to c$ as $y \to \infty$ then

$$S_R^K(x) = \int_0^\infty K(\tfrac{\alpha}{R})\,d\,A_0(\alpha)$$

exists for every R, and

$$S_R^K(x) \to c, \quad \text{as } R \to \infty,$$

for every x.

Remark: In stating assumption (4) we have presupposed $K(\alpha)$ to be differentiable (or rather an indefinite integral). However, it would suffice to assume

$$\int_0^\infty |dK(\alpha)| < \infty$$

without any assumption of differentiability, but the proof would become slightly more cumbersome.

Proof:

$$\int_0^y K(\tfrac{\alpha}{R})\,d\,A_0(\alpha)$$

$$= K(y/R)A_0(y) - \int_0^y \frac{1}{R}\,K'(\tfrac{\alpha}{R})A_0(\alpha)d$$

For fixed R, $K(\frac{\alpha}{R})$ has the same properties as $K(\alpha)$, and it can be easily verified from our assumptions that the right side tends to a limit as $y \to \infty$, and that

$$\int_0^\infty K(\tfrac{\alpha}{R})\,d\,A_0(R) = -\int_0^\infty \frac{1}{R}\,K'(\tfrac{\alpha}{R})A_0(\alpha)d$$

After replacing $A_o(y)$ by $A_o(y)- c$ we may assume that $c = o$. Choosing B sufficiently large and keeping it fixed, we obtain:

$$|\int_o^\infty K(\tfrac{\alpha}{R})dA_o(\alpha)| = |\int_o^\infty \tfrac{1}{R} K'(\tfrac{\alpha}{R})A_o(\alpha)d\alpha|$$

$$= |\int_o^B + \int_B^\infty|$$

$$= [O(1)\int_o^B \tfrac{1}{R}|K'(\tfrac{\alpha}{R})|d\alpha + o(1)\int_B^\infty \tfrac{1}{R}|K'(\tfrac{\alpha}{R})|d\alpha]$$

$$= [O(1)\int_o^{B/R} |K'(\alpha)|d\alpha + o(1)\int_o^\infty |K'(\alpha)|d\alpha]$$

$$= o(1),$$

as $R \to \infty$.

THEOREM 20. Assumptions: (1) $l \geq 1$, $K(\alpha)$ is twice differentiable in $o \leq \alpha < \infty$, (2) $K(o) = 1$, (3) $K(y) y^l \to o$ as $y \to \infty$, (4) $K'(y)y \to o$ as $y \to \infty$, (5) $\int_o^\infty \alpha |K''(\alpha)| d\alpha < \infty$, (6) $A_o(y) = o(y^l)$, $A_1(y) = \int_o^y A_o(\alpha)d\alpha = cy + o(y)$.

Conclusion:

$$\int_o^\infty K(\tfrac{\alpha}{R})d\, A_o(\alpha)$$

exists for every R and tends to c as $R \to \infty$.

Proof:

$$\int_o^y K(\tfrac{\alpha}{R})dA_o(\alpha) = K(y/R)A_o(y) - \int_o^y [K(\tfrac{\alpha}{R})]' A_o(\alpha)d\alpha$$

$$= K(y/R)A_o(y) - \int_o^y [K(\tfrac{\alpha}{R})]' d\, A_1(\alpha)$$

$$= K(y/R)A_o(y) - \tfrac{1}{R} K'(\tfrac{\alpha}{R})A_1(y) + \int_o^y \tfrac{1}{R^2} [K(\tfrac{\alpha}{R})]'' A_1(\alpha)d\alpha$$

Now, due to our assumptions, if $y \to \infty$, the right side exists, and in fact

$$\int_0^\infty K(\tfrac{\alpha}{R})d\, A_0(\alpha) = \int_0^\infty \frac{1}{R^2}\; K''(\tfrac{\alpha}{R})A_1(\alpha)d\alpha \;.$$

On replacing $A_0(\alpha)$ by $A_0(\alpha) - c$ we may again assume $c = 0$, and our conclusion follows from

$$\int_0^\infty \frac{1}{R^2}\, |K''(\tfrac{\alpha}{R})|\;|A_1(\alpha)|d\alpha \le \int_0^B + \int_B^\infty$$

$$= O(1)\int_0^B \frac{\alpha}{R^2}\,|K''(\tfrac{\alpha}{R})|d\alpha + o(1)\int_B^\infty \frac{\alpha}{R^2}|K''(\tfrac{\alpha}{R})|d\alpha \;.$$

§14. Abel and Gauss summability

Let

$$S_R = \int_0^\infty a(\alpha)e^{-\alpha/R}d\alpha \;;$$

$$T_R = \int_0^\infty a(\alpha)e^{-\alpha^2/R^2}d\alpha;$$

let

$$x = 1/R.$$

Theorem 21. Assumptions: (1) $a(\alpha)$ is L_1-integrable in every finite interval $0 \le \alpha \le R$, and

$$T(x) = \int_0^\infty a(\alpha)e^{-\alpha^2x^2}d\alpha$$

exists for every $x > 0$, (2) $T(x)$ is bounded, $|T(x)| \le M$, $0 < x < \infty$ (3) $\lim_{x\to+0} T(x) = c$, (4) $\int_0^\infty |a(\alpha)|e^{-\alpha x}d\alpha < \infty$ for

$0 < x < \infty$, *so that* $S(x) = \int_0^\infty a(\alpha)e^{-\alpha x}d\alpha$ *exists for* $0 < x < \infty$.

Conclusion:

$$\lim_{x \to +0} S(x) = c.$$

Remark: Assumptions (1) and (4) are certainly satisfied if $a(y) = O(y^k)$ for some $k > 0$.

Proof: We have:

$$\alpha > 0 \qquad e^{-\alpha} = \frac{1}{\pi}\int_0^\infty \frac{\cos \alpha x}{1+x^2}\, d\alpha$$

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)y}\, dy$$

$$e^{-\alpha} = \frac{2}{\pi}\int_{y=0}^\infty \int_{x=0}^\infty e^{-y} \cos \alpha x\, e^{-x^2 y}\, dy\, dx$$

$$e^{-\alpha} = \int_0^\infty \frac{e^{-y}}{\sqrt{\pi y}} \cdot e^{-\alpha^2/4y}\, dy$$

$$= \int_0^\infty \frac{e^{(-1/4)u^2}}{\sqrt{\pi}} \cdot \frac{e^{-\alpha^2 u^2}}{4u^2}\, du$$

$$= \int_0^\infty \gamma(u)e^{-\alpha^2 u^2}\, du, \quad \text{say},$$

where

$$\gamma(u) \geq 0 .$$

Setting $\alpha = 0$, we obtain

$$1 = \int_0^\infty \gamma(u)du$$

Now,

$$S(x) = \int_0^\infty a(\alpha)e^{-\alpha x}\, d\alpha$$

$$= \int_0^\infty \int_0^\infty a(\alpha)\gamma(u)e^{-\alpha^2 x^2 u^2} d\alpha\, du$$

$$= \int_0^\infty T(x\, u)\gamma(u)du,$$

by assumption (4) and $\gamma \geq 0$ and Fubini's theorem. Now $|T(x)| \leq M$ and hence by dominated convergence it is easy to see that $\lim_{x \to +0} S(x) = c$.

More generally, we have the following

THEOREM 22. *Assumptions*: (1) *Let* $K(\alpha)$, $\Delta(\alpha)$ *be defined for* $0 \leq \alpha < \infty$; $K(\alpha)$, $\Delta(\alpha)$ *are positive*; $K(0) = \Delta(0) = 1$; *there exists a relation*

$$K(\alpha) = \int_0^\infty \gamma(u)\Delta(\alpha\, u)du, \quad \gamma(u) \geq 0 \ ,$$

(2) $a(\alpha)$ *is* L_1*-integrable in every finite interval* $0 \leq \alpha < y$ *and*

$$T(x) = \int_0^\infty a(\alpha)\, \Delta(\alpha\, x)d\alpha$$

exists for every x, $0 < x < \infty$, (3) $|T(x)| \leq M$, $0 < x < \infty$, (4) $\lim_{x \to +0} T(x) = c$, (5) $\int_0^\infty |a(\alpha)|\, |K(\alpha\, x)|d\alpha < \infty$, $0 < x < \infty$ *so that*

$$S(x) = \int_0^\infty a(\alpha)K(\alpha\, x)d\alpha$$

exists for every x.

Conclusions:

$$\lim_{x \to +0} S(x) = c .$$

§15. Boundary values

Let $f(x) \in L_1$, $T[f] = \phi(\alpha)$, and let

$$f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\alpha) e^{-ix\alpha - y\alpha^2} \, d\alpha$$

(15*) $$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(z) \frac{1}{\sqrt{y}} e^{-\frac{(x-z)^2}{4y}} dz ,$$

the function $f(x,y)$ arising from $f_R(x)$ of theorem 15 on setting $\frac{1}{R^2} = y$.

Properties of $f(x,y)$:

(15.1) For every $y > 0$, $f(x,y) \in L_1$ as a function in x (Lemma 2).

(15.2) $\lim_{y \to +0} f(x,y) = f(x)$ in L_1-norm (th.14)

(15.3) For every $y > 0$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and belong to L_1.

For instance,

$$\sqrt{\pi} \frac{\partial f}{\partial x} = \int_{-\infty}^{\infty} f(z) \left\{ -\frac{(x-z)}{2y^{3/2}} e^{-\frac{(x-z)^2}{4y}} \right\} dz$$

and for fixed $y > 0$ this is

$$\int_{-\infty}^{\infty} f(z) h(x-z) dz$$

where $h(x) \in L_1$ and hence $\frac{\partial f}{\partial x} \in L_1$ by theorem 2; similarly for successive derivatives

$$\frac{\partial^{(r)} f}{\partial x^r}$$

of any integral order r.

(15.4) For every $y > 0$, there exists a function $\frac{\partial f}{\partial y} \in L_1$ such that

$$\lim_{h \to 0} \int_{-\infty}^{\infty} \left| \frac{f(x,y+h)-f(x,y)}{h} - \frac{\partial f}{\partial y} \right| dx = 0.$$

We mean that the literal partial derivative of $f(x,y)$ with respect to y is the limit in norm of the difference - quotient. For,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(z) y^{-1/2} e^{-\frac{(x-z)^2}{4y}} dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+z)\phi(y,z)dz,$$

where

$$\phi(y,z) = y^{-1/2} e^{-z^2/4y} .$$

Hence

$$\int_{-\infty}^{\infty} \left| \frac{f(x,y+h)-f(x,y)}{h} - \frac{\partial f}{\partial y} \right| dx$$

$$\leq \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x+z)| \, . \left| \frac{\phi(y+h,z)-\phi(y,z)}{h} - \frac{\partial \phi}{\partial y} \right| dz$$

$$\leq \|f\| \, . \int_{-\infty}^{\infty} \left| \frac{\phi(y+h,z)-\phi(y,z)}{h} - \frac{\partial \phi}{\partial y} \right| dz .$$

Now we know that

$$\lim_{h \to 0} \frac{\phi(y+h,z)-\phi(y,z)}{h} = \frac{\partial \phi}{\partial y} ;$$

to show that this relation holds in L_1-norm, it is enough to show that, for fixed $y > 0$,

$$\left| \frac{\phi(y+h,z)-\phi(y,z)}{h} - \frac{\partial\phi}{\partial y} \right| \leq \psi(z) \in L_1$$

uniformly in h. It suffices to show separately that

(15.41) $$\left| \frac{\partial\phi}{\partial y} \right| \leq \psi(z) \in L_1$$

and that

(15.42) $$\left| \frac{\phi(y+h,z)-\phi(y,z)}{h} \right| \leq \psi(z) \in L_1$$

uniformly in

(15.43) $$0 < y_0 - |h| \leq y \leq y_0 + |h| .$$

Now (15.41) follows directly from

(15.44) $$\frac{\partial\phi}{\partial y} = \frac{1}{2y^{3/2}} e^{-z^2/4y} - \frac{1}{4y^{3/2}} z^2 e^{-z^2/4y} .$$

Furthermore

$$\frac{\phi(y+h,z)-\phi(y,z)}{h} = \frac{1}{h} \int_0^h \phi_1(y+t,z)dt$$

where $\phi_1(y,z) = \frac{\partial\phi}{\partial y}$, and thus (15.42) follows from the fact that (15.41) holds uniformly in (15.43), as can again be verified from (15.44).

(15.5) For every $y > 0$, $\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial y}$ (by direct computation).

(15.6) $$\int_{-\infty}^{\infty} |f(x,y)|dx \leq M \ (= \int_{-\infty}^{\infty} |f(x)|dx)$$

(by Lemma 2).

<u>THEOREM 23</u>. <u>For given</u> $f(x) \in L_1$, <u>any</u> $f(x,y)$ <u>with properties</u> (15.1) - (15.6) <u>must be the function given by the formula</u> (15*).

Proof: By (15.1), for any $y > 0$, we can form the Fourier transform of $f(x,y)$; let it be $\phi(\alpha,y)$. We know that if $f_n \to f$ in L_1-norm, then $\phi_n(\alpha) \to \phi(\alpha)$ uniformly (see 1.5)). Hence, by property (15.2), $\phi(\alpha,y) \to \phi(\alpha)$, as $y \to +0$, for every α.

Again, from (15.3) and theorem 3,

$$T\left[\frac{\partial^2 f}{\partial x^2}\right] = (-i\alpha)^2 \phi(\alpha,y); \tag{15.7}$$

furthermore,

$$T\left[\frac{f(x,y+h)-f(x,y)}{h}\right] = \frac{\phi(\alpha,y+h)-\phi(\alpha,y)}{h}$$

and by property (15.4), there exists $\frac{\partial\phi(\alpha,y)}{\partial y}$ such that

$$\frac{\partial\phi(\alpha,y)}{\partial y} = \phi_{\frac{\partial f}{\partial y}} . \tag{15.8}$$

Using property (15.5), (15.7) and (15.8), and the uniqueness theorem for transforms, we obtain:

$$\frac{\partial\phi(\alpha,y)}{\partial y} = -\alpha^2\phi(\alpha,y).$$

Therefore

$$\phi(\alpha,y) = A(\alpha)e^{-\alpha^2 y}$$

for each α. We know that

$$\phi(\alpha,y) \to \phi(\alpha)$$

as $y \to 0$; hence $A(\alpha) = \phi(\alpha)$, so that

$$T[f(x,y)] = \phi(\alpha)e^{-\alpha^2 y} .$$

However, by theorem 2 the function given by (15*) has the same Fourier transform as $f(x,y)$; thus the two functions are equal almost everywhere, as asserted.

We will now have a similar discussion of the Abel kernel. For the remainder of this section, let $f(x) \in L_1$, $T[f] = \phi(\alpha)$ and

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\alpha)e^{-ix\alpha - y|\alpha|}\,d\alpha$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} f(x+z)\,\frac{y}{y^2+z^2}\,dz$$

(15**) $$= \frac{2}{\pi}\int_{-\infty}^{\infty} f(z)\,\frac{y}{y^2+(y-z)^2}\,dz .$$

It is then not hard to verify the following

<u>Properties of f(x,y)</u>:

(15.1)' $f(x,y) \in L_1$ as a function in x.

(15.2)' $f(x,y) \to f(x)$ in L_1-norm as $y \to +0$.

(15.3)' $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and belong to L_1.

(15.4)' $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ exist in the same manner as in (15.4) and

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) .$$

(15.5)' for every $y > 0$, $-\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2}$

(15.6)' $\int_{-\infty}^{\infty} |f(x,y)|dx \leq M \; (= \int_{\infty}^{\infty} |f(x)|dx)$.

THEOREM 24. *For given* $f(x) \in L_1$, *any* $f(x,y)$ *with properties* (15.1)' - (15.6)' *must be the function given by formula* (15**).

Proof: Let $T[f(x,y)] = \phi(\alpha,y)$. From (15.3)' and theorem 3, we have,

$$T[\frac{\partial^2 f}{\partial x^2}] = -\alpha^2 \phi(\alpha,y);$$

Since

$$T[\frac{f(x,y+h) - f(x,y)}{h}] = \frac{\phi(\alpha,y+h) - \phi(\alpha,y)}{h}$$

and because of (15.4)', there exist $\frac{\partial \phi(\alpha,y)}{\partial y}$ and $\frac{\partial^2 \phi(\alpha,y)}{\partial y^2}$

such that $$\frac{\partial \phi(\alpha,y)}{\partial y} = \phi_{\frac{\partial f}{\partial y}} .$$

Similarly

$$\frac{\partial^2 \phi(\alpha,y)}{\partial y^2} = \phi_{\frac{\partial^2 f}{\partial y^2}} .$$

Hence

$$\frac{\partial^2 \phi(\alpha,y)}{\partial y^2} = \alpha^2 \phi(\alpha,y)$$

so that

$$\phi(\alpha,y) = A(\alpha)e^{-\alpha y} + B(\alpha)e^{\alpha y} .$$

But, $\phi(\alpha,y) \rightarrow \phi(\alpha)$ as $y \rightarrow 0$; therefore $A(\alpha) + B(\alpha) = \phi(\alpha)$.

However, for all y, $|\phi_f(\alpha)| \leq \| f \| = M$, by (15.6)', that is

$$|\phi(\alpha,y)| \leq M ,$$

or

$$|B(\alpha)\, e^{\alpha y}| \leq M + |A(\alpha)\, e^{-\alpha y}|$$

$$\leq M + |A(\alpha)|$$

or

$$|B(\alpha)| \leq \frac{M + |A(\alpha)|}{e^{\alpha y}} .$$

Letting $y \to \infty$, we see that $B(\alpha) = 0$ for $\alpha > 0$, and similarly $A(\alpha) = 0$ for $\alpha < 0$. Hence

$$\phi(\alpha,y) = \phi(\alpha) .\ e^{-|\alpha| y} .$$

By the uniqueness theorem for transforms it follows that f(x,y) is the stated function (15**).

<u>Remarks</u>: The functions (15*) and (15**) are such that

(15.9) $$\lim_{y \to 0} f(x,y) = f(x)$$

almost everywhere, for each of them. However theorem 21 implies that at every point x at which (15.9) exists for (15**) it also exists for (15*), but perhaps not conversely. In other words, for a given boundary function f(x) in L_1 the solution (15*) of the heat-equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 ,$$

is more stable than the corresponding solution (15**) of the wave-equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0,$$

(with t written instead of the previous y).

§16. Mean values

Let $q > 0$, and let

$$S_R = \frac{1}{R^q} \int_0^\infty a(\alpha) K(\tfrac{\alpha}{R}) d\alpha ,$$

$$= \frac{1}{R^q} \int_0^\infty K(\tfrac{\alpha}{R}) dA(\alpha)$$

where

$$A(\alpha) = \int_0^\alpha a(x) dx$$

THEOREM 25. (1) $K(\alpha)$ is defined and absolutely continuous in $0 \leq \alpha < \infty$, (2) $K(y)y^q \to 0$ as $y \to \infty$, and $\int_0^\infty |K(\alpha)|\alpha^{q-1}\, d\alpha < \infty$; (3) $\int_0^\infty |K'(\alpha)|\alpha^q\, d\alpha < \infty$, (4) $A(y) = c.y^q + o(y^q)$ or $\lim_{t\to\infty} \frac{1}{t^q} \int_0^t a(\alpha) d\alpha = c$.

Conclusion: S_R exists for every R and

$$\lim_{R\to\infty} S_R = c.q \int_0^\infty K(\alpha)\alpha^{q-1}\, d\alpha.$$

Proof: Assumption (4) implies that $A(y) = O(y^q)$ as $y \to \infty$. Hence

$$\frac{1}{R^q} \int_0^y K(\tfrac{\alpha}{R}) dA(\alpha) = \frac{1}{R^q} K(y/R)A(y) - \int_0^y \frac{A(\alpha)}{R^{q+1}} K'(\tfrac{\alpha}{R}) d\alpha$$

so that

$$S_R = - \int_0^\infty K'(\tfrac{\alpha}{R}) \frac{A(\alpha)}{R^{q+1}}\, d\alpha = - \int_0^\infty \frac{K'(\alpha)A(\alpha R)}{R^q}\, d\alpha .$$

Now,

$$\frac{A(\alpha R)}{R^q} \to c\,\alpha^q\,, \text{ as } R \to \infty,$$

and

$$\frac{|A(\alpha R)|}{R^q} \leq \frac{d + c\,\alpha^q R^q}{R^q}$$

where d is a constant. Hence, by dominated convergence (see assumption (3)) we have

$$\lim_{R\to\infty} S_R = -c \int_0^\infty K'(\alpha)\alpha^q\, d\alpha$$

$$= -c \int_0^\infty \alpha^q\, dK(\alpha) = cq \int_0^\infty K(\alpha)\alpha^{q-1}\, d\alpha\,.$$

Example: Let $K(\alpha) = e^{-\alpha}$ and $q = 1$; then we see that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t a(\alpha)d\alpha = c$$

implies

$$\lim_{\varepsilon\to 0} \varepsilon\int_0^\infty a(\alpha)e^{-\varepsilon\alpha}\, d\alpha = c$$

that is, (C,1) summability (Fejer's) implies Abel summability.

THEOREM 26. Assumptions: (1) $K(\alpha)$ is measurable in $(0,\infty)$ and $|K(\alpha)| \leq c\, K_0(\alpha)$ where $K_0(\alpha)$ satisfies conditions (1) - (3) of theorem 25, (2) $A(y) = cy^q + o(y^q)$, (3) $\int_0^y |a(\alpha)|d\alpha = O(y^q)$.

Conclusion:

$$\lim_{R\to\infty} S_R = cq \int_0^\infty K(\alpha)\alpha^{q-1}\, d\alpha\,.$$

The proof may be constructed on the same lines as in th.25.

<u>Examples</u>: If $K(\alpha) = (\frac{\sin\alpha}{\alpha})^2$ and $K_0(\alpha) = \frac{c}{1+\alpha^2}$

then theorem 26 applies, and we have the following:

if $a(\alpha) \geq 0$, then $\lim_{t\to\infty} \frac{1}{t} \int_0^t a(\alpha)d\alpha = 0$

implies

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\infty a(\alpha) \frac{\sin^2\alpha\varepsilon}{\alpha^2} d\alpha = 0.$$

If $a(\alpha) = c\, e^{ix\alpha}$, $x \neq 0$, then the mean value of $a(\alpha)$, say, $M\{a(\alpha)\}$ satisfies:

$$M\{a(\alpha)\} = \lim_{t\to\infty} \frac{1}{t} \int_0^t a(\alpha)d\alpha = 0;$$

if $x = 0$, then $M\{a(\alpha)\} = c$. If

$$a(\alpha) = \sum_{m=0}^{n} c_m e^{ix_m\alpha}, \text{ where } x_0 = 0 \text{ and } x_m \neq 0,\ m > 0,$$

then

$$c_0 = M\{a(\alpha)\} = \lim_{\varepsilon\to 0} \int_0^\infty a(\alpha)e^{-\varepsilon\alpha}\, d\alpha \quad .$$

More generally, if

$$a(\alpha) \sim \sum c_n\, e^{ix_n\alpha}$$

is an almost periodic function in the original definition of H. Bohr, say, then the Fourier coefficient c_n which is computable as

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T a(\alpha)e^{-ix_n\alpha} d\alpha$$

is also computable as

$$c_n = \lim_{\epsilon \to 0} \int_0^\infty a(\alpha) e^{-(\epsilon + i x_n \alpha)} d\alpha$$

the limit existing.

§17. Tauberian theorems

THEOREM 27. Assumptions: (1) $q > 0$, $a(\alpha) \geq 0$, (2) $K(\alpha)$ is measurable, and $\int_0^\infty |K(\alpha)| \alpha^{q-1} d\alpha < \infty$, (3) $S(R) = \frac{1}{R^q} \cdot \int_0^\infty a(\alpha) K(\frac{\alpha}{R}) d\alpha$ exists absolutely for all R, (4) $\lim_{R \to \infty} S(R) = c.q \int_0^\infty K(\alpha) \alpha^{q-1} d\alpha$.

Conclusion: if $\lambda_1 > 0, \ldots, \lambda_n > 0$; $c_1, \ldots, c_n$ are constants, and if

$$\Lambda(\alpha) = c_1 K(\alpha \lambda_1) + \ldots + c_n K(\alpha \lambda_n)$$

and

$$\sigma(R) = \frac{1}{R^q} \int_0^\infty a(\alpha) \, \Lambda(\tfrac{\alpha}{R}) d\alpha ,$$

($\sigma(R)$ existing absolutely), then

$$\lim_{R \to \infty} \sigma(R) = c.q \int_0^\infty \Lambda(\alpha) \alpha^{q-1} d\alpha.$$

Proof: For fixed λ

$$\frac{1}{R^q} \int_0^\infty a(\alpha) K(\frac{\alpha \lambda}{R}) d\alpha$$

$$= \frac{1}{\lambda^q} (\frac{\lambda}{R})^q \int_0^\infty a(\alpha) K(\alpha . \frac{\lambda}{R}) d\alpha$$

$$\to c . \frac{1}{\lambda^q} . q \int_0^\infty K(\alpha) \alpha^{q-1} d\alpha$$

$$= c.q \int_0^\infty K(\alpha\lambda)\alpha^{q-1}d\alpha .$$

Remark: The class of such "polynomials" in $K(\alpha)$ as $\sum_{n=1}^{m} c_n K(\alpha\lambda_n)$, $\lambda_n > 0$, will be denoted by $\{\Lambda(\alpha)\}$, and individual members of the latter class will be denoted by $\Lambda^1, \Lambda^2, \ldots$, etc.

THEOREM 28. If $\Delta(\alpha)$ has the following properties: namely, given $\varepsilon > 0$, there exist $\Lambda^1(\alpha)$, $\Lambda^2(\alpha)$ belonging to $\{\Lambda(\alpha)\}$ such that

$$\Lambda^1(\alpha) < \Delta(\alpha) < \Lambda^2(\alpha)$$

in $0 \leq \alpha < \infty$, and

$$q \int_0^\infty [\Lambda^2(\alpha) - \Lambda^1(\alpha)]\, \alpha^{q-1}\, d\alpha < \varepsilon$$

then

$$\lim_{R\to\infty} \sigma(R) = c.q \int_0^\infty \Delta(\alpha)\alpha^{q-1}\, d\alpha$$

where $\sigma(R)$ is defined as:

$$\sigma(R) = \frac{1}{R^q} \int_0^\infty a(\alpha)\Delta\left(\frac{\alpha}{R}\right)d\alpha ,$$

the latter integral existing absolutely.

Proof: Since $a(\alpha)$ is non-negative,

$$\sigma^1(R) \leq \sigma(R) \leq \sigma^2(R)$$

where

$$\sigma^{1,2}(R) = \frac{1}{R^q} \int_0^\infty a(\alpha)\; \Lambda^{1,2}(\alpha)d\alpha .$$

Hence

$$c.q \int_0^\infty \Lambda^1(\alpha)\alpha^{q-1}d\alpha = \lim \sigma^1(R) \leq \underline{\lim}\, \sigma(R) \leq \overline{\lim}\, \sigma(R)$$

$$\leq \lim \sigma^2(R) = c.q \int_0^\infty \Lambda_2(\alpha)\alpha^{q-1}d\alpha \quad .$$

Now σ^1 and σ^2 tend to finite limits as $R\to\infty$ on account of theorem 27, and

$$c\ q \int_0^\infty \Lambda^2(\alpha)\alpha^{q-1}\ d\alpha$$

$$= c\ q[\ \int_0^\infty \Delta(\alpha)\alpha^{q-1}\ d\alpha + \int_0^\infty [\ \Lambda^2(\alpha) - \Delta(\alpha)]\alpha^{q-1}\ d\alpha\ ]$$

$$\leq c\ q \int_0^\infty \Delta(\alpha)\alpha^{q-1}\ d\alpha + \varepsilon \quad .$$

Similarly,

$$c\ q \int_0^\infty \Lambda^1(\alpha)\alpha^{q-1}\ d\alpha \geq c\ q \int_0^\infty \Delta(\alpha)\alpha^{q-1}\ d\alpha - \varepsilon \quad .$$

Hence

$$-\varepsilon + c\ q \int_0^\infty \Delta(\alpha)\alpha^{q-1}\ d\alpha$$

$$\leq \underline{\lim}\, \sigma(R) \leq \overline{\lim}\, \sigma(R) \leq c\ q \int_0^\infty \Delta(\alpha)\alpha^{q-1}\ d\alpha + \varepsilon \quad .$$

Therefore

$$\lim_{R\to\infty} \sigma(R) = c\ q \int_0^\infty \Delta(\alpha)\alpha^{q-1}\ d\alpha \ .$$

<u>Remark</u>: The set of functions $\{\Delta(\alpha)\}$ obtained from $K(\alpha)$ by the above process may be called the <u>Karamata extension</u> of $K(\alpha)$ and we will denote it by $E\{K(\alpha)\}$.

<u>Lemma 3</u>: Let $K(\alpha) = e^{-\alpha}$. Suppose $f(\alpha)$ is such that to each $\varepsilon > 0$ there corresponds a $\Lambda_\varepsilon(\alpha)$ such that $|f(\alpha) - \Lambda_\varepsilon(\alpha)| < \varepsilon e^{-\alpha}$. Then, $F \in E\{K(\alpha)\}$.

Proof: By hypothesis, $\Lambda_\varepsilon(\alpha)$ is of the form $\sum_{n=1}^{m} c_n K(\alpha\lambda_n)$ where $\lambda_n > 0$ and $K(\alpha) = e^{-\alpha}$.

If we choose

$$\Lambda^1(\alpha) = \Lambda_\varepsilon(\alpha) - \varepsilon\, e^{-\alpha}$$

and

$$\Lambda^2(\alpha) = \Lambda_\varepsilon(\alpha) + \varepsilon\, e^{-\alpha}$$

then we have

$$\Lambda^1(\alpha) < f(\alpha) < \Lambda^2(\alpha)$$

and

$$q\int_0^\infty (\Lambda^2 - \Lambda^1)\alpha^{q-1}\, d\alpha = \varepsilon . 2q\int_0^\infty e^{-\alpha} d\alpha = o(1)$$

as $\varepsilon \to 0$. Hence, by theorem 28,

$$f(\alpha) \in E\{K(\alpha)\} .$$

<u>Remarks</u>: If $K(\alpha) = e^{-\alpha}$, then every function $f(\alpha)$ which is continuous in $0 \leq \alpha < \infty$ and vanishes outside a finite interval belongs to $E\{e^{-\alpha}\}$. For, let $g(\alpha) = f(\alpha).e^{\alpha}$; then $g(\alpha)$ is a continuous function of α in $0 \leq \alpha < \infty$. Writing x for $e^{-\alpha}$ we obtain the function $g(\log \frac{1}{x})$ which is continuous in $0 \leq x \leq 1$. By Weierstrass' theorem it can be approximated to by polynomials, so that

$$|g(\log \tfrac{1}{x}) - \sum_{m=1}^{n} c_m x^m| < \varepsilon , \quad 0 \leq x < 1 .$$

Reverting to α,

$$|g(\alpha) - \sum_{m=1}^{n} c_m e^{-m\alpha}| < \varepsilon$$

or,

$$|f(\alpha) - \sum_{m=1}^{n} c_m e^{-(m+1)\alpha}| < e^{-\alpha};$$

hence from lemma 3 our statement follows. We have thus obtained a sufficient condition for $f(\alpha) \in E\{e^{-\alpha}\}$. We shall now give necessary and sufficient conditions for $f(\alpha) \in E\{e^{-\alpha}\}$. For simplicity, we suppose $q = 1$.

<u>THEOREM 29</u>. <u>In order that</u> $f(\alpha) \in E\{e^{-\alpha}\}$ <u>it is necessary and sufficient that</u> (1) $f(\alpha)$ <u>be Riemann integrable in every finite interval</u> $0 \leq \alpha \leq A$, <u>and</u> (2) $|f(\alpha)| \leq c\, e^{-\lambda_0 \alpha}$ <u>for some</u> $\lambda_0 > 0$ <u>in the entire interval</u> $0 \leq \alpha < \infty$.

Proof: $E\{K(\alpha)\}$ was defined to be the class of all functions $f(\alpha)$ which satisfy the pair of conditions:

$$\Lambda^1(\alpha) < f(\alpha) < \Lambda^2(\alpha) \tag{17.1}$$

and

$$\int_0^\infty (\Lambda^2 - \Lambda^1)\, d\alpha < \varepsilon \tag{17.2}$$

where $\Lambda^2, \Lambda^1 \in \{\Lambda(\alpha)\}$ and $\Lambda(\alpha) = \sum_{n=1}^{m} c_n e^{-\lambda_n \alpha}$, $\lambda_n > 0$. (Notice that every $\lambda_n > 0$ and not merely ≥ 0). The conditions (17.1) and (17.2) imply that for a fixed A, the function $f(\alpha)$ is Riemann integrable in $0 \leq \alpha \leq A$, and since

$$|f(\alpha)| \leq |\Lambda^1| + |\Lambda^2|$$

it follows that

$$|f(\alpha)| < c\, e^{-\lambda_0 \alpha}, \quad \lambda_0 > 0$$

and the necessity-part of the theorem follows.

To prove that the conditions are sufficient, we observe:

$$(17.3) \qquad |f(\alpha)\, e^{\frac{\lambda_0\alpha}{2}}| \leqslant c\, e^{-\frac{\lambda_0\alpha}{2}} \to 0, \quad \text{as } \alpha \to \infty .$$

Setting $e^{-\frac{\lambda_0\alpha}{2}} = x$ and $f(\alpha)e^{\frac{\lambda_0\alpha}{2}} = F(x)x = g(x)$ we observe that $g(x)$ is bounded and Riemann integrable in the closed interval $0 \leqslant x \leqslant 1$ (the point $x = 1$ is included because of (17.3)). Hence it is possible to determine polynomials $p(x)$ and $P(x)$ such that

$$p(x) < g(x) < P(x)$$

and

$$\int_0^1 |P(x) - p(x)|\,dx < \varepsilon .$$

That is

$$p(e^{-\frac{\lambda_0\alpha}{2}}) < f(\alpha)e^{\frac{\lambda_0\alpha}{2}} < P(e^{-\frac{\lambda_0\alpha}{2}})$$

and

$$\int_0^\infty |P(e^{-\frac{\lambda_0\alpha}{2}}) - p(e^{-\frac{\lambda_0\alpha}{2}})|\, e^{-\frac{\lambda_0\alpha}{2}}\, d\alpha < \varepsilon .$$

Hence,

$$\Lambda^1 < f(\alpha) < \Lambda^2 ,$$

and

$$\int_0^\infty (\Lambda^2 - \Lambda^1)\, d\alpha < \varepsilon$$

if only we choose

$$\Lambda^1(\alpha) = p(e^{-\frac{\lambda_0\alpha}{2}})\, e^{-\frac{\lambda_0\alpha}{2}}$$

and

$$\Lambda^2(\alpha) = P(e^{-\frac{\lambda_0\alpha}{2}})\, e^{-\frac{\lambda_0\alpha}{2}}$$

The factor $e^{-\frac{\lambda_0\alpha}{2}}$ ensures that the constant term of the polynomial p is multiplied by $e^{-\lambda_0\alpha}$ where $\lambda_0 > 0$; this is necessary for $\lambda^1(\alpha)$.

Remark: Let

$$f(x) = \begin{cases} 0, & \text{for } 1 < \alpha < \infty , \\ 1, & \text{for } 0 \leq \alpha \leq 1 . \end{cases}$$

then $f(\alpha) \in E\{e^{-\alpha}\}$. Hence, in theorem 28 we can substitute $f(\alpha)$ for $\Delta(\alpha)$ and $e^{-\alpha}$ for $K(\alpha)$ thus obtaining

THEOREM 30. If $a(\alpha) \geq 0$ and

$$S(R) = \frac{1}{R} \int_0^\infty a(\alpha) e^{-\alpha/R}\, d\alpha$$

exists absolutely for all $R > 0$, and $\lim_{R\to\infty} S(R) = c$, then

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R a(\alpha) d\alpha = c .$$

Remarks: (1) Theorem 30 states that a non-negative function which is Abel-summable is Fejer-summable. This is the conditioned converse of an easier result (theorem 25).

(2) Instead of $a(\alpha) \geq 0$ we may suppose $a(\alpha) \geq -M$, $M \geq 0$, for we have then only to consider the function $a(\alpha) + M$ which is non-negative. It follows that for a function bounded on one side, Abel and Fejer summations are equivalent.

CHAPTER II

FOURIER TRANSFORMS IN L_1 (SEVERAL VARIABLES)

§1. Riemann Lebesgue Lemma; Composition; Convolution

Let $f(x_1,\ldots,x_k)$ be of class L_1 over E_k -- the space defined by $-\infty < x_r < \infty$, $r = 1,\ldots,k$. Let $\phi(\alpha_1,\ldots,\alpha_k)$, the α's being real, be defined by

$$\phi(\alpha_1,\ldots,\alpha_k) = \int_{E_k} f(x_1,\ldots,x_k)e^{i(\alpha_1 x_1+\ldots+\alpha_k x_k)}dx_1\ldots dx_k$$

Then $\phi(\alpha_1,\ldots,\alpha_k)$ is called the Fourier transform of $f(x_1,\ldots,x_k)$. We sometimes denote the function and its transform by $f(x)$ and $\phi(\alpha)$ respectively.

THEOREM 31. If $f(x_1,\ldots,x_k) \in L_1$ in E_k, then $\phi(\alpha_1,\ldots,\alpha_k)$ exists, and is bounded for all α's and

$$\lim_{\alpha_1^2+\ldots+\alpha_k^2\to\infty} \phi(\alpha_1,\ldots,\alpha_k) = 0.$$

Proof: By hypothesis,

$$\int_{E_k} |f(x)|dV_k \leq M ,$$

where dV_k denotes the k-dimensional volume-element.

And,

$$|\phi(\alpha)| \leq \int_{E_k} |f(x)||e^{i\sum\alpha_r x_r}|dV_k \leq M .$$

Therefore $\phi(\alpha)$ exists and is bounded for all α's.

We shall prove the limit-theorem as before, for an everywhere dense subset of the class of L_1-functions - - namely for step-functions. Let $g(x) = 1$ in the k-dimensional box $a_i \leq x_i \leq b_i$ and $g(x) = 0$ outside the box. Let $\psi(\alpha) = Tg$. Then

$$\psi(\alpha) = \int_{E_k} g(x) e^{i\sum \alpha_r x_r} dV$$

$$= \int_{a_1}^{b_1} e^{i\alpha_1 x_1}\, dx_1 \ldots \int_{a_k}^{b_k} e^{i\alpha_k x_k}\, dx_k \,.$$

Because of the Riemann-Lebesgue Lemma for one variable, each of the factors is bounded and tends to zero as $\sum \alpha_r^2 \to \infty$; as a matter of fact, if $\alpha_r \to \infty$ for a single r, then $\psi(\alpha) \to 0$; the convergence is uniform in the α's; hence $\psi(\alpha) \to 0$ for box-functions, hence also for step-functions. The extension to all L_1-functions follows as in Chapter I, since every function in $L_1(E_k)$ is an L_1-limit of step-functions of the kind just introduced.

<u>THEOREM 32</u>. <u>If</u> $Tf = \phi$ <u>and</u> $Tg = \psi$ <u>where</u> $f, g \in L_1$, <u>then</u>

$$\int_{E_k} f(x)\psi(x) dV_x = \int_{E_k} g(y)\phi(y) dV_y$$

The proof is similar to the one-variable case.

<u>THEOREM 33</u>. <u>If</u> $f, g \in L_1$, <u>then their</u> (<u>resultant, or</u>) <u>convolution</u> $h(x)$ <u>which is defined as</u>

$$\int_{E_k} f(x-y) g(y) dV_y$$

<u>exists for almost all</u> x, <u>belongs to</u> L_1, <u>and</u>

$$T[h(x)] = \phi(\alpha)\cdot\psi(\alpha) .$$

Here again the proof is similar to the one-dimensional case; Fubini's theorem in k-variables has to be used.

§2. Uniqueness theorem

In the case of one variable, we proved that two L_1-functions having the same Fourier transform are equal almost everywhere; we shall now give a similar proof for several variables. Before so doing, we have to make one observation.

If

$$f(x_1, \ldots, x_k) = \prod_r f_r(x_r)$$

and each $f_r(x_r)$ belongs to L_1 in $(-\infty, \infty)$ then

$$\phi(\alpha_1, \ldots, \alpha_k) = \prod_r \phi_r(\alpha_r)$$

where

$$\phi_r(\alpha_r) = \int_{-\infty}^{\infty} f_r(x_r) e^{i\alpha_r x_r}\, dx_r .$$

<u>THEOREM 34</u>. <u>If</u> $f(x_1, \ldots, x_k) \in L_1$ <u>and</u> $\phi(\alpha_1, \ldots, \alpha_k) = 0$ <u>then</u> $f(x) = 0$ <u>almost everywhere</u>.

Proof: Let $g^{\varepsilon}_{a_j, b_j}(x)$ be the one-variable function defined as follows:

$$g^{\varepsilon}_{a_j,b_j}(x) = \begin{cases} 1, & a_j \leq x \leq b_j \\ 0, & x > b_j+\varepsilon,\ x < a_j-\varepsilon \\ \dfrac{b_j-x+\varepsilon}{\varepsilon}, & b_j \leq x \leq b_j+\varepsilon \\ \dfrac{x+a_j+\varepsilon}{\varepsilon}, & a_j-\varepsilon \leq x \leq a_j . \end{cases}$$

Let

$$T[g^{\varepsilon}_{a_j,b_j}(x_j)] = \psi^{\varepsilon}(\alpha_j).$$

As we have observed in Chapter I, §6,

$$\psi^{\varepsilon}(\alpha_j) = 0(\frac{1}{\alpha_j^2}), \quad \text{as } \alpha_j \to \infty$$

and is bounded in E_1, and so

(2.1) $$\psi^{\varepsilon}(\alpha_j) \in L_1 \text{ in } -\infty < \alpha_j < \infty ;$$

hence

(2.2) $$g^{\varepsilon}_{a_j,b_j}(x_j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^{\varepsilon}(\alpha_j) e^{-i\alpha_j x_j} d\alpha_j .$$

Now define

(2.3) $$g^{\varepsilon}_{I}(x_1,\ldots,x_k) = \prod_{j=1}^{k} g^{\varepsilon}_{a_j,b_j}(x_j),$$

where I is the previously introduced box, and thus g^{ε}_{I} vanishes outside a larger box whose position is defined by I and ε. Let the Fourier transform of $g^{\varepsilon}_{I}(x_1,\ldots,x_k)$ be $\psi^{\varepsilon}_{I}(\alpha_1,\ldots,\alpha_k)$. By the observation just preceding this theorem,

(2.4) $$\psi^{\varepsilon}_{I}(\alpha_1,\ldots,\alpha_k) = \prod_{j=1}^{k} \psi^{\varepsilon}(\alpha_j) .$$

Substituting (2.2) in (2.3) and then using (2.4), we obtain

(2.5) $$g_I^{\varepsilon}(x) = \frac{1}{(2\pi)^k} \int_{E_k} \psi_I^{\varepsilon}(\alpha) e^{-i\sum \alpha_j x_j} dV_{\alpha}$$

where $\psi_I^{\varepsilon}(\alpha)$ belongs to L_1 in E_k on account of (2.1). Now, owing to the absolute convergence of the integral in (2.5) we deduce that

$$\int_{E_k} f(y) g_I^{\varepsilon}(x-y) dV_y = \int_{E_k} \phi(\alpha) \psi_I^{\varepsilon}(\alpha) e^{i\sum \alpha_j x_j} dV_{\alpha}$$

Since by assumption, $\phi(\alpha) = 0$, we have:

$$\int_{E_k} f(y) g_I^{\varepsilon}(x-y) dV_y = 0 .$$

If we fix a_j, b_j and let $\varepsilon \to 0$ it follows that

$$\int_I f(y) dV_y = 0$$

for every box I; but this implies $f(x) = 0$ almost everywhere.

§3. Gauss summability formula

Let

$$g(x_1, \ldots, x_k) = e^{-\varepsilon^2 (x_1^2 + \ldots + x_k^2)}$$

$$= \prod_{r=1}^{k} e^{-\varepsilon^2 x_r^2}$$

and

$$T[g(x_1, \ldots, x_k)] = \phi(\alpha_1, \ldots, \alpha_k).$$

We have seen that

$$\phi(\alpha_1, \ldots, \alpha_k) = \prod_{r=1}^{k} \phi(\alpha_r) ,$$

where

$$\phi(\alpha_r) = \int_{-\infty}^{\infty} e^{-\varepsilon^2 x_r^2} \cdot e^{i\alpha_r x_r} \, dx_r$$

$$= \frac{1}{\varepsilon} \pi^{1/2} e^{-\alpha_r^2/4\varepsilon^2}$$

by (7.10)Chapter I. Hence

$$\phi(\alpha_1, \ldots, \alpha_k) = \frac{\pi^{k/2}}{\varepsilon^k} e^{-(\alpha_1^2 + \ldots + \alpha_k^2/4\varepsilon^2)}.$$

Setting $\frac{1}{\varepsilon} = R$, we have

$$(3.1) \qquad T[e^{-\sum x_j^2/R^2}] = R^k \pi^{k/2} e^{-R^2(\sum \alpha_j^2)/4} .$$

Define

$$(3.2) \quad \overset{G}{S}_R(x) = \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha_1, \ldots, \alpha_k) e^{-i\sum \alpha_j x_j} \cdot e^{-\sum \alpha_j^2/R^2} dV_\alpha$$

By the composition theorem (Th.32)and (3.1), we have:

$$(3.3) \qquad \overset{G}{S}_R(x) = \frac{\pi^{k/2}}{(2\pi)^k} R^k \int_{E_k} f(x+y) e^{-R^2(\sum y_j^2)/4} .$$

Also,

$$(3.4) \quad \overset{G}{S}_R(x) - f(x) = \frac{R^k}{2^k \pi^{k/2}} \int_{E_k} \{f(x+y) - f(x)\} e^{-R^2(\sum y_j^2)/4} dV_y$$

since

$$\frac{\pi^{k/2} R^k}{(2\pi)^k} \int_{E_k} \cdot e^{-R^2(\sum y_j^2)/4} dV_y = \frac{\pi^{k/2} R^k}{(2\pi)^k} \int_0^\infty t^{k-1} w_{k-1} e^{-R^2 t^2/4} dt$$

where

$$w_{k-1} = \frac{2\pi^{k/2}}{\Gamma(k/2)}$$

is the k-1 dimensional volume element of the unit-sphere: $y_1^2 + \ldots + y_k^2 = 1$ and so

$$\frac{\pi^{k/2}R^k}{(2\pi)^k}\int_{E_k} e^{-R^2(\sum y_j^2)/4}dV_y = \frac{R^k}{2^{k-1}\Gamma(k/2)}\int_0^\infty t^{k-1}e^{-R^2t^2/4}dt$$

$$= \frac{2}{\Gamma(k/2)}\int_0^\infty x^{k-1}e^{-x^2}dx$$

$$= 1 .$$

Next, define

$$(3.5) \qquad f_x(t) = \frac{1}{w_{k-1}}\int_\sigma f(x+ty)d\sigma_y$$

where σ is the unit-sphere $y_1^2 + \ldots + y_k^2 = 1$, $d\sigma$ is its (k-1) dimensional volume element, and w_{k-1} is its (k-1) dimensional volume. Furthermore, set

$$(3.6) \qquad g_x(t) = t^{k-1}[f_x(t) - f(x)].$$

Then, since (3.4) can be written as

$$S_R^G(x) - f(x) = \frac{R^k w_{k-1}}{2^k\pi^{k/2}}\int_0^\infty t^{k-1}e^{-R^2t^2/4}dt \int_\sigma [f(x+ty)-f(x)]d\sigma_y$$

we have

$$(3.7) \qquad S_R^G(x)-f(x) = \frac{R^k}{2^{k-1}\Gamma(k/2)}\int_0^\infty e^{-R^2t^2/4}\, g_x(t)dt \ .$$

§4. Gauss summability theorem

THEOREM 35. *If* $f(x_1,\ldots,x_k) \in L_1$ *in* E_k *and* $g_x(t)$ *is defined as in* (3.6), *and if* $H(t)$, $0 \leq t < \infty$ *is such that* $t^k H(t)$ *decreases monotonely to zero, thus also implying* $H(t) \downarrow 0$ *and* $H(t) = O(\frac{1}{t^{k+\varepsilon}})$ *as* $t \to \infty$, *then for a fixed* x, *the condition*

$$G_x(t) = \int_0^t g_x(y)dy = o(t^k) \quad \text{as } t \to 0$$

implies

$$R^k \int_0^\infty g_x(t)H(Rt)dt = o(1), \quad \text{as } R \to \infty .$$

THEOREM 36. *If* $f(x_1,\ldots,x_k)$ *satisfies the assumptions of theorem 35 and* H *is such that there exists an* H_0 *with the property* $|H(t)| \leq H_0(t)$ *and* H_0 *now satisfies the assumptions on* H *in theorem 35, then the condition*

$$\int_0^t |g_x(t)|dt = o(t^k) \quad \text{as } t \to 0$$

implies

$$R^k \int_0^\infty g_x(t)H(Rt)dt = o(1) \quad \text{as } R \to \infty.$$

The proofs of theorems 35 and 36 are very similar to those of theorems 6 and 7.

Actually the conditions imposed on $H(t)$ are at present somewhat more special than those imposed previously, and this accounts for a certain amount of simplification in the wording of the present theorems, if not their proofs.

Using these theorems and the formula (3.7) we deduce that

$$S_R^G(x) \to f(x) \qquad \text{as } R \to \infty$$

whenever

$$\int_0^t g_x(t)dt = o(t^k) \qquad \text{as } t \to 0.$$

Now if $f(x) \in L_1$ this latter condition is satisfied almost everywhere; it is equivalent with stating that

$$(4.1) \qquad f(x_0) = \lim_{\varepsilon \to 0} \frac{1}{V(S_\varepsilon)} \int_{S_\varepsilon} f(x)dV_x ,$$

where S_ε is a sphere of radius ε with center at x_0, and $V(S_\varepsilon)$ is the volume of S_ε. The validity of (4.1) for almost all x_0 in E_k is then part of the Lebesgue-Vitali theorem. Hence we have

THEOREM 37. *The Fourier transform of a Lebesgue integrable function is Gauss-summable almost everywhere. More explicitly:*

$$\lim_{R\to\infty} \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha_1, \ldots, \alpha_k) e^{-i\sum \alpha_j x_j} \cdot e^{-\sum \alpha_j^2/R^2} dV_\alpha = f(x)$$

almost everywhere.

§5. *Application of summability theorem; inversion formula:*

THEOREM 38. *If* $f(x_1, \ldots, x_k) \in L_1$ *and* $\phi(\alpha_1, \ldots, \alpha_k) \in L_1$, *in* E_k, *then*

$$f(x_1, \ldots, x_k) = \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha_1, \ldots, \alpha_k) e^{-i\sum \alpha_j x_j} dV_\alpha$$

almost everywhere.

Proof: Since

$$\int_{E_k} |\phi(\alpha)| dV_\alpha < \infty$$

it follows that

$$\int_{E_k} |\phi(\alpha) e^{-i\sum \alpha_j x_j}| dV_\alpha < \infty$$

and by similarity with the one-dimensional case (Th.8) we deduce this theorem from theorem 37.

THEOREM 39. *If* $f(x_1, \ldots, x_k) \in L_1$ *and* $|f(x)| \leq M$ *and* $\phi(\alpha_1, \ldots, \alpha_k) \geq 0$, *then*

$$f(x_1, \ldots, x_k) = \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha_1, \ldots, \alpha_k) e^{-i\sum \alpha_j x_j} dV_\alpha$$

almost everywhere.

Proof: As in theorem 9, we first deduce from the hypothesis that $\phi(\alpha_1, \ldots, \alpha_k) \in L_1$ and application of theorem 38 gives the result.

Remarks: The function

$$f_0(x) = \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha) e^{-i\sum \alpha_j x_j} dV_\alpha$$

itself is continuous. Hence f(x) differs from a continuous function in a null set. If f(x) is assumed not only to belong to L_1 but also to be continuous, then theorems 38 and 39 assert that the inversion formula holds *everywhere*.

§6. Norms, Continuity, Parseval relations.

If $f(x_1 \ldots, x_k) \in L_1$, set

$$w(h_1, \ldots, h_k) = w(h) = \int_{E_k} |f(x+h) - f(x)| dV_x .$$

Then as in §9, Chapter I., it is possible to see that $w(h) \longrightarrow 0$ as $|h| \to 0$, where $|h| = (h_1^2 + \ldots + h_k^2)^{1/2}$ (see Th.10). The general case has again to be reduced to "elementary functions" of the type $w_I(x)$ which is equal to 1 in the k-dimensional box denoted by I and vanishes outside. Using this fact, we observe that if $f(x) \in L_1$, and both f and g are bounded, then their convolution exists for every x, is bounded, continuous and of class L_1. The proof is similar to that of theorem 11. Hence it follows, as in theorem 12 and 13, that if f(x) and g(x) are bounded functions of class L_1, then

$$\int_{E_k} f(x+y)\overline{g(y)} dV_y = \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha)\overline{\psi(\alpha)}\, e^{-i\sum \alpha_j x_j} dV_\alpha$$

at <u>every</u> point x, of E_k, and in particular, we have:

$$\int_{E_k} |f(y)|^2 dV_y = \frac{1}{(2\pi)^k} \int_{E_k} |\phi(\alpha)|^2 dV_\alpha$$

the integrals being finite.

§7. Radial functions

We shall now show that if f(x) in E_k depends only on the distance of (x) from the origin, namely, $\sqrt{x_1^2 + \ldots + x_k^2}$, then $\phi(\alpha)$ also depends only on the distance $\sqrt{\alpha_1^2 + \ldots + \alpha_k^2}$; we call such functions f(x) <u>radial functions</u>. Our object is to prove that for radial functions, the Fourier transform in several variables goes over into the "Bessel transform" in one variable. In dealing with the class L_p, however, the functions are taken only almost everywhere in E_k, and radiality has therefore to be defined with some precision.

A function $f(x_1,\ldots,x_k)$ defined in E_k presents a radial element of $L_p(E_k)$, $p \geq 1$, or is said to be radial, if except for a null set in E_k, we have the relation $f(x_1,\ldots,x_k) = F(\sqrt{x_1^2+\ldots+x_k^2})$ the positive square root being taken, where $F(t)$ is defined everywhere in $0 \leq t < \infty$. It follows that a function $f(x_1,\ldots,x_k)$ or an element of $L_p(E_k)$, is radial if and only if it is invariant under all orthogonal transformations relative to the origin.

We claim that such radial elements form a closed subset of $L_p(E_k)$, meaning that if $\{F_n(x_1^2+\ldots+x_k^2)^{1/2}\}$ converge in $L_p(E_k)$-norm towards a function $f(x_1,\ldots,x_k)$, then the limit-function, by alteration on a null-set in E_k, may be assumed to be again of the form $F(\sqrt{x_1^2+\ldots+x_k^2})$. Working with Cauchy-sequences, the L_p-convergence of $\{F_n(\sqrt{x_1^2+\ldots+x_k^2})\}$ amounts to

$$(7.1) \qquad \lim_{m,n\to\infty} \int_0^\infty t^{k-1}|F_m(t)-F_n(t)|^p dt = 0 \;;$$

because $f_n(x_1,\ldots,x_k)$ belongs to L_p in E_k if and only if $t^{\frac{k-1}{p}} F_n(t) \in L_p$ in $0 \leq t < \infty$ owing to the fact that

$$\int_{E_k} |f_n(x_1,\ldots,x_k)|^p dV_x = w_{k-1} \int_0^\infty |F_n(t)|^p t^{k-1} dt .$$

Now, from (7.1) it follows that $\{t^{\frac{k-1}{p}} F_n(t)\}$ is a Cauchy-sequence in $L_p(0,\infty)$, which has therefore a limit in $L_p(0,\infty)$, which we choose to write as $t^{k-1/p} F(t)$ and the function $F(t)$ thus obtained has the desired property.

If $k = 1$, a radial function is no other than an even function.

THEOREM 40. *If* $f(x_1,\ldots,x_k) \in L_1$ *in* E_k *depends only on* $R = \sqrt{x_1^2+\ldots x_k^2}$, *that is,* $f(x_1,\ldots,x_k) = g(R)$, *then the function*

$$\phi(\alpha_1,\ldots,\alpha_k) = \int_{E_k} f(x) e^{i\sum \alpha_r x_r} dV_x$$

depends only on $\alpha = \sqrt{\alpha_1^2+\ldots+\alpha_k^2}$. *We can then express* ϕ *as*

$$\phi(\alpha) = \frac{(2\pi)^{k/2}}{\alpha^{\frac{1}{2}(k-2)}} \int_0^\infty g(R) R^{k/2} J_{\frac{k-2}{2}}(\alpha R) dR .$$

If, further, $\alpha_1^2+\ldots+\alpha_k^2 = s$, *then*

(a) *for* $k = 2m + 2$,

$$\phi(\alpha) = (-1)^m \pi^{m+1} 2^{2m+1} \frac{d^m}{ds^m} \int_0^\infty g(R) R\, J_0(s^{\frac{1}{2}} R) dR$$

and

(b) *for* $k = 2m + 1$,

$$\phi(\alpha) = (-1)^m \pi^m 2^{2m+1} \frac{d^m}{ds^m} \int_0^\infty g(R) \cos(s^{\frac{1}{2}} R) dR,$$

where $J_\mu(x)$ *denotes the Bessel function of the first kind,*

<u>of</u> <u>order</u> μ.

We give two proofs of this theorem.

First Proof: For given (α_j) we subject the integral $\phi(\alpha_1,\ldots,\alpha_k)$ to an orthogonal transformation

$$y_r = \sum_s b_{rs}\, x_s$$

with determinant $+1$, and

$$b_{11} : b_{12} : \ldots : b_{1k} = \alpha_1 : \alpha_2 : \ldots : \alpha_k .$$

We then have

$$\sum x_r^2 = \sum y_r^2 , \quad \sum \alpha_r x_r = (\sum \alpha_r^2)^{\frac{1}{2}} \cdot y_1 .$$

Hence,

$$(7.2)\quad \phi(\alpha_1,\ldots,\alpha_k) = \int_{E_k}\cdots\int g(\sqrt{y_1^2+\ldots+y_k^2})\, e^{-i\sqrt{\sum \alpha_r^2}\, y_1}\, dV_y$$

so that $\phi(\alpha_1,\ldots,\alpha_k)$ depends only on α.

Next suppose that $k \geq 2$. Then,

$$\phi(\alpha) = \int_{-\infty}^{\infty} e^{-i\alpha y_1} dy_1 \int_{E_{k-1}} g(\sqrt{y_1^2+\ldots+y_k^2})\, dV_y$$

where V_y is now the $(k-1)$-dimensional volume-element. Hence

$$\phi(\alpha) = w_{k-2}\int_{-\infty}^{\infty} e^{-i\alpha y_1} dy_1 \int_0^{\infty} g(\sqrt{y_1^2+z^2})\, z^{k-2} dz .$$

This integral may be considered over the half-plane $z \geq 0$ in the (y,z) plane. Using polar coordinates, $y_1 = r\cos\theta$ and $z = r\sin\theta$, we observe that θ runs from 0 to π, while r runs from 0 to ∞. Hence,

$$\phi(\alpha) = w_{k-2} \int_0^\infty g(r) r^{k-1} K(\alpha r) dr,$$

where

$$K(s) = \int_0^\pi e^{-s \cos \theta} (\sin \theta)^{k-2} \, d\theta$$

$$= \frac{J_{\frac{k-2}{2}}(s)}{s^{\frac{k-2}{2}}} \Gamma(\tfrac{k-1}{2}) \Gamma(\tfrac{1}{2}) 2^{\frac{k-2}{2}} ,$$

from which our result follows.

It holds also for k = 1, because

$$J_{-\frac{1}{2}}(t) = \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} \cos t$$

Since

$$\frac{J_p(x)}{x^p} = -\frac{1}{x} \frac{d}{dx} \left(\frac{J_{p-1}(x)}{x^{p-1}}\right)$$

we have, on setting $x = s^{\frac{1}{2}} \cdot R$,

$$\frac{J_p(\alpha R)}{\alpha^p} = \left(\frac{-2}{R}\right)^m \frac{d^m}{ds^m} \left(\frac{J_{p-m}(s^{\frac{1}{2}}R)}{(\sqrt{s})^{p-m}}\right)$$

from which we get results (a) and (b) of theorem 40 by setting $p = \frac{k-2}{2}$.

Second Proof:

$$\phi(\alpha_1, \ldots, \alpha_k) = \int_{E_k} f(x_1, \ldots, x_k) e^{i(\alpha_1 x_1 + \cdots + \alpha_k x_k)} dV_x .$$

Set

$$x_j = \sum_{\chi=1}^{k} a_{j\chi} \, y_\chi , \qquad j = 1, \ldots, k$$

and

$$D = |a_{jl}| \neq 0 .$$

Denote by T the transformation $(x) \to (y)$. Then

$$\phi(\alpha_1, \ldots, \alpha_k) = D \int_{E_k} f(Ty) e^{i(\alpha . Ty)} dV_y$$

$$= D \int_{E_k} f(Ty) e^{i(T^*\alpha . y)} dV_y$$

where $(\alpha . Ty)$ denotes the inner product of the vectors α and Ty, and T^* is the transpose of T.

If $f(Ty) = f(y)$, then it follows that $\phi(\alpha) = D.\phi(T^*\alpha)$. If T is orthogonal, then $T^* = T^{-1}$ and $D = +1$, hence $\phi(b) = \phi(Tb)$. Thus, if a function is radial, its transform is also radial.

Now let $f(x_1, \ldots, x_k) = F(x_1^2 + \ldots + x_k^2)^{\frac{1}{2}}$ and let $\phi(\alpha_1, \ldots, \alpha_k)$ denote the transform of f. Then $\phi(\alpha_1, \ldots, \alpha_k) = \psi(\alpha, 0, \ldots, 0)$, where $\alpha = \sqrt{\alpha_1^2 + \ldots + \alpha_k^2}$; hence

$$\psi(\alpha) = \int_{E_k} F(\sqrt{x_1^2 + \ldots + x_k^2}) e^{i\sum \alpha_r x_r} dx_1 \ldots dx_k$$

(7.3) $$= \int_0^\infty F(t) t^{k-1} dt \int_\sigma e^{i\sum \alpha_r x_r} dw_{k-1} ,$$

where dw_{k-1} is the $(k-1)$-dimensional volume (or surface) of the k-dimensional unit-sphere, and σ denotes the sphere: $\sum x_r^2 = t^2$.

It is easily seen that

$$(7.4) \qquad \int_\sigma e^{i\sum \alpha_r x_r} dw_{k-1} = c \int_0^\pi e^{i\alpha t \cos\theta} (\sin\theta)^{k-2} d\theta$$

where c is a constant. Set

$$(7.5) \qquad S(r) = c\int_0^\pi e^{ir\cos\theta}(\sin\theta)^{k-2} d\theta \; .$$

Then it follows from (7.3) that

$$(7.6) \qquad \psi(\alpha) = \int_0^\infty F(t)t^{k-1}S(\alpha t)dt \; .$$

Our object now is to evaluate S(r); if we consider the integral (7.5) for complex values of r, then S(r) is an entire function of exponential type:

$$(7.7) \qquad |S(r)| \leq e^{|r|} \; .$$

Hence S(r) has a power series expansion, say,

$$S(r) = \sum_{n=0}^\infty a_n r^n \; .$$

If we now choose $f(x_1, \ldots, x_k)$ as the special function

$$e^{-(x_1{}^2 + \ldots + x_k{}^2)}$$

so that $F(t) = e^{-t^2}$, we obtain

$$\psi(\alpha) = \pi^{k/2} e^{-\alpha^2/4} \; . \qquad \text{(see (3.1))}.$$

Thus, as a consequence of (7.6), we have the relation

$$\pi^{k/2} e^{-\alpha^2/4} = \int_0^\infty e^{-t^2} t^{k-1} S(\alpha t)dt$$

$$= \sum_{n=0}^{\infty} a_n \alpha^n \int_0^\infty e^{-t^2} t^{k+n-1} dt$$

(7.8)
$$= \sum_{n=0}^{\infty} \frac{1}{2} \Gamma\left(\frac{k+n}{2}\right) a_n \alpha^n \quad ,$$

the interchange of integration and summation being justified by (7.7). Expanding $e^{-\alpha^2/4}$ as a power-series, and comparing coefficients, we obtain from (7.8):

$$S(\alpha t) = 2\pi^{k/2} \sum_{n=0}^{\infty} \frac{(\alpha t)^{2n} (-1)^n}{4^n \Gamma(n) \Gamma(\mu+n-1)} , \quad \mu > -\tfrac{1}{2} \quad .$$

Since

$$J_\mu(t) = \left(\frac{t}{2}\right)^\mu \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^{2n} \Gamma(n) \Gamma(\mu+n+1)} , \ \mu > -\frac{1}{2}$$

we obtain:

(7.9)
$$S(\alpha t) = (2\pi)^{k/2} \frac{J_{\frac{k-2}{2}}(\alpha t)}{(\alpha t)^{\frac{k-2}{2}}} \quad .$$

Using (7.9) in (7.6), we obtain

$$\psi(\alpha) = \frac{(2\pi)^{k/2}}{\alpha^{\frac{k-2}{2}}} \int_0^\infty F(t) t^{k/2} J_{\frac{k-2}{2}}(\alpha t) dt$$

from which our theorem follows.

Using theorem 40 and the inversion-formula in several variables obtained in theorem 38, we can state the following

<u>THEOREM 41</u>. *If* $f(t)$ *is continuous in* $0 \leq t < \infty$, *and* $\int_0^\infty |f(t)| t^{k-1} dt < \infty$, *and if*

$$\phi(\alpha) = (2\pi)^{k/2} \int_0^\infty f(t) t^{k-1} V_{\frac{k-2}{2}}(\alpha t) dt$$

where $V_k(x) = J_k(x)/x^k$, *is such that*

$$\int_0^\infty |\phi(\alpha)| \alpha^{k-1} d\alpha < \infty,$$

then we have, for every t,

$$f(t) = \frac{1}{(2\pi)^{k/2}} \int_0^\infty \phi(\alpha) \alpha^{k-1} V_{\frac{k-2}{2}}(\alpha t) d\alpha .$$

<u>Examples</u>. If $k = 1$, then $J_{-\frac{1}{2}}(t) = (\frac{2}{\pi t})^{\frac{1}{2}} \cos t$ and hence we have

$$\phi(\alpha) = (\frac{2}{\pi})^{\frac{1}{2}} \int_0^\infty f(t).(\frac{2}{\pi t})^{\frac{1}{2}} \cos t\alpha \, dt$$

(7.10) $$= 2\int_0^\infty f(t) \cos t\alpha . dt ,$$

which is the ordinary cosine-transform of $f(t)$.

If $k = 3$, $J_{\frac{1}{2}}(t) = (\frac{2}{\pi t})^{1/2} \sin t$, and hence we have

(7.11) $$\phi(\alpha) = (2\pi)^{\frac{3}{2}} \int_0^\infty f(t) t^2 (\frac{2}{\pi})^{\frac{1}{2}} \frac{\sin t\alpha}{t\alpha} dt$$

$$\alpha \, \phi(\alpha) = 4\pi \int_0^\infty f(t) t \sin t\alpha \, dt .$$

Setting $\Phi(\alpha) = \alpha \phi(\alpha)$ and $F(t) = t f(t)$ we get:

(7.12) $$\Phi(\alpha) = 4\pi \int_0^\infty F(t) \sin \alpha t \, dt.$$

Inverting (7.11) we get

(7.13) $$f(t) = \frac{1}{2\pi^2} \int_0^\infty \phi(\alpha)\alpha^2 \frac{\sin t\alpha}{t\alpha} d\alpha$$

and hence

$$F(t) = \frac{1}{2\pi^2} \int_0^\infty \Phi(\alpha) \sin t\alpha \, d\alpha .$$

Since $k = 3$, (7.11) and (7.13) will hold if

(7.15) $$\int_0^\infty |f(t)| t^2 dt < \infty , \quad \int_0^\infty |\phi(\alpha)| \alpha^2 d\alpha < \infty .$$

Remark: On the other hand, it would seem that we can interpret (7.12) and (7.14) as sine-transforms in one variable, and hence that these are valid if only

(7.16) $$\int_0^\infty |f(t)| t \, dt < \infty \text{ and } \int_0^\infty |\phi(\alpha)| \alpha \, d\alpha < \infty .$$

There is, however, no discrepancy between (7.15) and (7.16), because we could interpret (7.12) as a sine-transform only if $F(t)$ were given as an odd function on $(-\infty, \infty)$, which is not the case here, because our $F(t)$ is defined only for $0 \leq t < \infty$.

§8. General summability for radial functions

In §3 of this chapter, we established the formula (3.7):

(8.1) $$S_R^G(x) - f(x) = \frac{1}{2^{k-1}\,\Gamma(k/2)} \int_0^\infty e^{-R^2t^2/4} g_x(t) dt$$

where

$$S_R^G(x) = \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha_1, \ldots, \alpha_k) e^{-i\sum \alpha_j x_j} . e^{-\sum \alpha_j^2/R^2} dV_\alpha$$

and

$$g_x(t) = t^{k-1}[f_x(t) - f(x)],$$

where

$$f_x(t) = \frac{1}{w_{k-1}} \int_\sigma f(x+ty) d\sigma_y .$$

It is clear that in the proof of (8.1) the significant feature of the convergence-factor

$$K(x) = e^{-(x_1^2 + \ldots + x_k^2)}$$

is its radiality; we use the fact that its transform is radial, $K(x) \in L_1$ in E_k, $K(0) = 1$, $K(x)$ is continuous at $x = 0$, and that if $H(\alpha)$ is the Fourier transform of $K(\alpha)$, then

$$\frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} H(\alpha) dV_\alpha = 1 .$$

Given an unspecified kernel $K(x)$ satisfying the conditions just enumerated, we can first of all establish the more general formula,

(8.2) $$S_R^K(x) - f(x) = \frac{1}{2^{k-1}\Gamma(k/2)\pi^{k/2}} \int_0^\infty R^k g_x(t) H(Rt) dt,$$

and if now we use theorems 35 and 36 we can determine sufficient conditions to ensure that $S_R^K(x) \to f(x)$ as $R \to \infty$, where

$$(8.3)\quad \overset{K}{\underset{R}{S}}(x) = \frac{1}{(2\pi)^k}\int_{E_k}\phi(\alpha_1,\ldots,\alpha_k)e^{-i\sum\alpha_j x_j}K\left(\frac{(\alpha_1^2+\ldots+\alpha_k^2)^{\frac{1}{2}}}{R}\right)dV_\alpha$$

<u>Example</u>. From (8.3) it follows that

$$(8.4)\qquad \overset{K}{\underset{R}{S}}(x) = \int_{E_k} f(x+y)R^k\, H(R(y_1{}^2+\ldots+y_k{}^2)^{\frac{1}{2}})dV_y$$

either by composition or convolution. Now take

$$K\left(\frac{(\alpha_1{}^2+\ldots+\alpha_k{}^2)^{\frac{1}{2}}}{R}\right) = e^{-\frac{1}{R}(\alpha_1{}^2+\ldots+\alpha_k{}^2)^{\frac{1}{2}}} \quad ; \ \frac{1}{R} = \varepsilon \ .$$

It is known that

$$e^{-\alpha} = \int_0^\infty \frac{e^{-y}}{(\pi y)^{1/2}} \cdot e^{-\alpha^2/4y}\, dy, \quad \text{(see 4, Chr.I.)}$$

so that

$$e^{-u\alpha} = \frac{u}{\sqrt{\pi}}\int_0^\infty e^{-u^2x-\alpha^2/4x} \cdot x^{-\frac{1}{2}}\, dx.$$

Hence

$$\int_{E_k} e^{-u(\alpha_1{}^2+\ldots+\alpha_k{}^2)^{\frac{1}{2}} + i\sum\alpha_j x_j}\, dV_\alpha$$

$$= \int_0^\infty e^{-u^2 y} \cdot y^{-\frac{1}{2}}\, u\, \pi^{-\frac{1}{2}} dy \int_{E_k} e^{-\frac{\alpha_1{}^2+\ldots+\alpha_k{}^2}{4y}} \cdot e^{i\sum\alpha_j x_j}\, dV_\alpha$$

$$= \int_0^\infty e^{-u^2 y}\, y^{-\frac{1}{2}}\, u\, \pi^{-\frac{1}{2}}\, e^{-y\sum\alpha_j{}^2}\, y^{\frac{k}{2}}\, \pi^{\frac{k}{2}}\, dy$$

$$= \int_0^\infty e^{-u^2y-\sum \alpha_j^2 y} \, y^{\frac{k-1}{2}} \pi^{\frac{k-1}{2}} u \, dy$$

$$= u\pi^{\frac{k-1}{2}} \int_0^\infty e^{-(u^2+\sum \alpha_j^2)y} \, y^{\frac{k-1}{2}} \, dy$$

$$(8.5) \qquad = \frac{u \, \pi^{\frac{k-1}{2}} \, \Gamma(\frac{k+1}{2})}{(u^2+\sum \alpha_j^2)^{\frac{k+1}{2}}} \, .$$

Hence, using the inversion-formula, we obtain

$$(8.6) \quad e^{-u(\alpha_1^2+\ldots+\alpha_k^2)^{\frac{1}{2}}} = \frac{\Gamma(\frac{k+1}{2})}{2^k \pi^{\frac{k+1}{2}}} \int_{E_k} \frac{u \, dV_x}{(u^2+x_1^2+\ldots+x_k^2)^{\frac{k+1}{2}}}$$

Again, using (8.5) in (8.3) and (8.4), we have:

$$(8.7) \quad \frac{1}{(2\pi)^k} \int_{E_k} \phi(\alpha_1, \ldots, \alpha_k) e^{-\varepsilon(\alpha_1^2+\ldots+\alpha_k^2)^{\frac{1}{2}}+i\sum \alpha_j x_j} dV_\alpha$$

$$= \frac{\Gamma(\frac{k+1}{2})\varepsilon}{\pi^{\frac{k+1}{2}}} \int_{E_k} \frac{f(x+y)dV_y}{(\varepsilon^2+ y_1^2+\ldots+ y_k^2)^{\frac{k+1}{2}}}$$

Formulas (8.6) and (8.7) are what we wanted to establish.

CHAPTER III

L_p-SPACES.

§1. Metric spaces.

A non-void set E is called a metric space if to every pair (x,y) of its elements, there corresponds a real number d(x,y) called the metric or distance function, which satisfies the following postulates:

(i) $d(x,y) > 0$ if $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

A sequence of elements $\{x_n\}$ belonging to E is called a Cauchy sequence if

$$\lim_{m,n\to\infty} d(x_m,x_n) = 0$$

The sequence is said to be convergent if there exists an element x of E such that $\lim_{n\to\infty} d(x_n,x) = 0$. x is called the limit of $\{x_n\}$, and we say that $\{x_n\}$ converges to the limit x; symbolically, $\lim_{n\to\infty} x_n = x$. It is easy to see that the limit is unique. While every convergent sequence is a Cauchy-sequence, the converse is not necessarily true.

If every Cauchy-sequence $\{x_n\}$ of E is necessarily convergent, then E is called <u>complete</u>.

If the space E is such that every one of its infinite sequences has a convergent subsequence, then E is said to be <u>compact</u>.

A compact metric space is necessarily complete. An element x_o is called a <u>point of accumulation</u> of the set $S \subset E$, if there exists a sequence of elements $\{x_n\}$ of S such that (i) not all $x_n = x_o$, (ii) and such that $\lim_{n \to \infty} x_n = x_o$.

The set S' of all accumulation points of S is called the <u>derived set</u> of S. The union of S and S' is called the <u>closure</u> of S and denoted by $\overline{S}$. If $S' \subset S$, then S is called a <u>closed</u> set. S is called <u>open</u> if its complement (E - S) is closed.

A set S is <u>dense</u> (or everywhere dense) in the space E if $\overline{S} = E$. The space E is separable if it contains a dense enumerable set.

Every compact metric space is separable. Every subspace of a separable space is separable.

§2. <u>Completion of a metric space</u>

Before we proceed further, it is desirable to prove the result that given a metric space E, it can be completed into a space F such that E is metrically isomorphic (or, isometric) with a subset G of F, with G dense in F; also, F is uniquely determined up to a metric isomorphism.

Suppose E is a metric space, and $X = \{x_n\}$, $Y = \{y_n\}$ are two Cauchy-sequences from E. Consider the sequence of distances $d_n = d(x_n, y_n)$ where d is the distance-function in E; this sequence tends to a limit as $n \to \infty$, because

$$|d(x_n,y_n) - d(x_m,y_m)|$$

$$\leq |d(x_n,y_n) - d(x_n,y_m)| + |d(x_n,y_m) - d(x_m,y_m)|$$

$$\leq d(y_n,y_m) + d(x_n,x_m) \to 0, \quad \text{as } m,n \to \infty;$$

We denote the limit by d(X,Y) and define the distance between the sequences X and Y to be d(X,Y). It is easily seen that d(X,Y) satisfies the requirements (ii) and (iii) of a metric.

In order to secure (i), we define two Cauchy-sequences S and T to be equivalent, $S \sim T$, if $d(S,T) = 0$. This relation $\sim$ is reflexive, symmetric and transitive; and hence it partitions E into disjoint classes P,Q,R..... of equivalent Cauchy sequences, such that two Cauchy sequences are equivalent or not according as they fall into the same class or different classes. It is also very easy to see that d(X,Y) is unaltered if we replace X,Y by equivalent sequences. Hence the metric is invariant under equivalence. If we now write d(P,Q) instead of d(X,Y), and if we introduce the space F of all equivalence classes P,Q,... of E, then d(P,Q) satisfies postulate (i) as well.

We shall now show that F contains a subset G which is metrically isomorphic with E, that G is dense in F and that

F is complete.

If any class P contains a <u>convergent</u> Cauchy sequence $\{X = \{x_n\}\}$ tending to the limit x, we call it a principal class. If all the terms of a sequence are identical with one another, we call it a principal sequence. (X = x,x,...) A principal class will then contain a principal sequence, namely, that sequence which is obtained by repitition of the limit of the convergent sequence which the principal class contains. Conversely any class containing a principal sequence (x) will contain all Cauchy sequences convergeing to x and is therefore a principal class. If P,Q are two principal classes containing respectively the principal sequence (x,x,x,.....) and (y,y,y,.....) it is clear that the distance d(P,Q) is the same as the distance d(x,y) in E. <u>Thus the subset G consisting of the principal classes of F is isometric with E</u>.

Let P be any element of F and $X = (x_n)$ be a member of P. Let $X^r=(x_r,x_r,\ldots)$ and let P^r be the principal class containing X^r, r = 1,2,... . Since X is a Cauchy sequence in E, we have $d(p,P^r) = d(X,X^r) \to 0$ as $r \to \infty$. Hence P is the limit (in F) of the convergent sequence (P^r) of principal classes; <u>this proves that G is dense in F</u>.

Let $(Q_1,Q_2,\ldots\ldots)$ be any Cauchy sequence in F. From the above paragraph we observe that any class Q_r is the limit of a convergent sequence of principal classes. $Q_r = \lim\limits_{k\to\infty} P_r^{\,k}$. Hence, there exist principal classes P_r,

$r = 1,2,3,\ldots$ such that $d_F(Q_r, P_r) < \frac{1}{r}$. Since $d_F(Q_r, P_r) \to 0$ as $r \to \infty$, and since $\{Q_r\}$ is a Cauchy sequence we see that

$$d(P_r, P_s) \leq d(P_r, Q_r) + d(Q_r, Q_s) + d(P_s, Q_s) \to 0$$

as $r, s \to \infty$, and hence $\{P_r\}$ is a Cauchy sequence of principal classes. However, $\{P_r\}$ converges to a limit P. For, denote by $X^r = (x_r, x_r, \ldots)$ the unique principal sequence contained in the principal class P_r. Then

$$d(x_m, x_n) = d(X^m, X^n) = d(P_m, P_n) \to 0$$

as $m,n \to \infty$, since $\{P_r\}$ is a Cauchy sequence. If P is the class containing X, it follows that

$$d(P, P_r) = d(X, X^r) \to 0$$

as $r \to \infty$. Thus $\{P_r\}$ converges to P. Since $d_F(Q_r, P_r) \to 0$ as $r \to \infty$, it follows that $\{Q_r\}$ also converges to P. Hence F is complete.

To prove that the extension F is unique upto a metric isomorphism we have only to note that if E is isometric with a dense subset G_1 of a complete metric space F_1 and also isometric with a dense subset G_2 of another complete metric space F_2, then this establishes an isometry between G_1 and G_2; however, for any complete metric spaces F_1, F_2 an isometry between subsets G_1, G_2 can be uniquely extended to an isometry between their closures $\overline{G}_1, \overline{G}_2$ and in our present case these closures are the spaces F_1, F_2 themselves. Thus we have the following

THEOREM 42. <u>If E is a given metric space (which is not complete), there exists a complete metric space F such that E is isometric with a dense subset G of F; and F is unique upto a metric isomorphism.</u>

§3. Banach spaces

A non-empty set of elements E is called a <u>linear</u> space (or vector space) if (i) it is an Abelian group with respect to an operation of addition (+), (ii) it admits multiplication (.) by real or complex numbers, this multiplication being associative and doubly distributive with respect to addition, and (iii) $1.x = x$ where $x \in E$.

A linear space E is said to be <u>normed</u> if there exists a real-valued function, called the <u>norm</u>, defined over E and satisfying the following requirements: if $x \in E$ and $\|x\|$ is the norm of x, then

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|x + y\| \leq \|x\| + \|y\|$

(3) $\|ax\| = |a| . \|x\|$, for every number a.

If we now define the distance between two elements x,y of E as $d(x,y) = \|x - y\|$ then $d(x,y)$ satisfies all the requirements of a metric and the normed linear space E becomes a metric space; therefore E is complete (according to the definition of a complete metric space in §1 of this Chr.) if whenever the sequence $\{x_n\}$ of E satisfies the condition

$$\lim_{m,n\to\infty} \| x_m - x_n \| = 0 ,$$

there exists an element x of E such that

$$\lim_{m\to\infty} \| x - x_m \| = 0 .$$

A complete normed linear space is called a Banach space.

THEOREM 43. Given a normed linear space N, there exists a smallest (upto isomorphism) completion to a Banach space B; if viewed as a metric space only, the space B is the previously constructed (Th.42) metric completion of N. In other words, if a metric space N derives its metric from a norm, then the metric completion of N is also its Banach completion.

Proof. In the space N the metric is: $d_N(x,y) = \| x - y \|_N$. By theorem 42, there exists a smallest space B such that B is again metric and complete and N is isometric with a subset of B. The space B consists of equivalent classes P,Q,.... of Cauchy sequences from N. If $X = \{x_n\}$ is contained in P and $Y = \{y_n\}$ in Q, then we define the class a.P as the class of sequences equivalent to the sequence $\{a\, x_n\}$ and $a\, P \oplus b\, Q$ as the class equivalent to $\{a\, x_n + b\, y_n\}$, where a and b are numbers. Note that $a\, x_n$ and $b\, y_n$ are elements of N, because the space is linear. This operation $\oplus$ of addition and multiplication by real numbers a,b,... , make B a <u>linear space</u>.

In theorem 42, the metric of B was defined as: $d_B(P,Q) = \lim_{n\to\infty} d_N(x_n, y_n)$ where $X = \{x_n\}$ and $Y = \{y_n\}$ are members of P and Q respectively. If we introduce an evaluation of elements in B as follows:

$$\| P \|_B = d_B(P,0) = \lim_{n\to\infty} d_N(x_n, 0),$$

then the function, $\| \ \|_B$, satisfies the requirements of a norm, and hence B is a <u>normed linear space</u>. Being metrically complete, it is a Banach space.

§4. Linear operations

Given two spaces S and S', if to each element of S there corresponds an element of S', such a correspondence is called an <u>operation</u> from S to S' or, a <u>transformation</u> of S into S'. S may be called the <u>domain-space</u> and S' the <u>range-space</u>; these two may coincide. If the <u>range-space</u> is the space of real or complex numbers, the operation is called a <u>functional</u>.

Let the Banach space B_f be mapped into the Banach space B_ϕ by the relation:

$$\phi = T f, \quad \phi \in B_\phi, \quad f \in B_f .$$

The operation Tf, or the operator T, is <u>linear</u> if

$$T(c_1 f_1 + c_2 f_2) = c_1 . Tf_1 + c_2 . Tf_2$$

for any numbers c_1, c_2 and any two elements f_1, f_2 of B_f. When B_ϕ is the space of real or complex numbers, Tf is a linear functional.

A transformation T of B_f into B_ϕ is closed if, whenever the sequences $\{f_n\}$ and $\{Tf_n\}$ exist, and converge to the limits f and g respectively, then f is in the domain of T and Tf = g.

A linear transformation T of B_f into B_ϕ is continuous at f if, whenever $\lim_{n\to\infty} \| f_n - f \| = 0$, we have $\lim_{n\to\infty} \| Tf_n - Tf \| = 0$; the transformation is simply called continuous, if it is continuous at every point of the domain B_f. The transformation is called bounded if there exists a number $M > 0$ such that $\| Tf \| \leq M \| f \|$ for every $f \in B_f$. If Tf is a bounded linear transformation, and $f_1, f_2 \in B_f$ then $\| Tf_1 - Tf_2 \| \leq M \| f_1 - f_2 \|$ (in case $B_f = B_\phi =$ the space of real numbers, this is equivalent to a Lipschitz condition). It is easy to see that a bounded linear trasnformation is continuous. For, the hypothesis implies that

$$\| T(f_n - f) \| = \| Tf_n - f \| \leq M \| f_n - f \|.$$

Conversely, if Tf is a linear transformation which is continuous at a single point g of B_f then it is bounded, and hence is continuous everywhere. For if Tf is not bounded, there exists a sequence f_n of elements of B_f such that

$$\| Tf_n \| > M_n \| f_n \| ,$$

where $M_n \to \infty$. If we set

$$g_n = g + \frac{f_n}{M_n \| f_n \|} ,$$

we have

$$\| g_n - g \| = \frac{1}{M_n} \to 0 \ , \quad \text{as } n \to \infty .$$

Since T is continuous at g, this would imply that $\| Tg_n - Tg \| \to 0$ as $n \to \infty$, which is impossible since

$$\| Tg_n - Tg \| = \frac{\| Tf_n \|}{M_n \| f_n \|} > 1 \ ;$$

thus we have the following

THEOREM 44. *For linear transformations the concepts of boundedness and continuity are equivalent.*

The bound of a linear operator T is defined to be

$$|T| = \underset{f \in B_f}{\text{l.u.b}} \frac{\| Tf \|}{\| f \|}$$

We next prove an important theorem on the convergence of a sequence of linear operators.

THEOREM 45. *If $\{T_n\}$ is a sequence of continuous linear operators defining the transformations $\phi_n = T_n f$ of the space B_f into B_ϕ, and if $|T_n|$ is uniformly bounded (that is, $|T_n| = M_n < M$ for all n), and if further, $T_n g$ converges to an operation Tg on a linear subset $\{g\}$ which is everywhere dense in B_f, then for every f in B_f the sequence $T_n f$ converges to an operation Tf and T is a linear continuous operator such that $|T| \leq M$.*

Proof. If f and g are arbitrary elements of B_f, we have

$$\| T_n f - T_m f \| \leq \| T_n f - T_n g \| + \| T_n g - T_m g \| + \| T_m g - T_m f \| .$$

Since $\{T_n\}$ converges on $\{g\}$ we have, by Cauchy-criterion, $\lim_{m,n\to\infty} \| T_n g - T_m g \| = 0$. Futhermore

$$\| T_n f - T_n g \| = \| T_n (f-g) \| \leq M_n \| f-g \| \leq M \| f-g \|$$

and similarly

$$\| T_m g - T_m f \| \leq M \| f-g \| .$$

Hence

$$\overline{\lim_{m,n\to\infty}} \| T_n f - T_m f \| \leq 2M \| f-g \| .$$

Since $\{g\}$ is everywhere dense in B_f, $\| f-g \|$ can be made as small as we please. Therefore

$$\overline{\lim_{m,n\to\infty}} \| T_n f - T_m f \| = 0 .$$

Since B_ϕ is complete, the sequence $T_n f$ has a limit Tf in B_ϕ. Now the operation Tf is linear, because

$$T(af) = \lim_{n\to\infty} T_n(af) = a \cdot \lim_{n\to\infty} T_n f = a \cdot Tf ,$$

and similarly $T(f_1 + f_2) = Tf_1 + Tf_2$. Finally, Tf has the bound M, for, given $\epsilon > 0$, if n is large enough we have

$$\begin{aligned} \| Tf \| &\leq \| Tf - T_n f \| + \| T_n f \| \\ &\leq \epsilon + M_n \| f \| \\ &\leq \epsilon + M \| f \| . \end{aligned}$$

As a corollary of the above result, we obtain the following

THEOREM 46. *If*

(i) $f(x) \in L_1(0,\infty)$ *and* S_α *is a set of numbers,*

(ii) $K(\alpha,t)$ *is Lebesgue measurable and uniformly bounded for* $\alpha \in S_\alpha$ *and* $0 \leq t < \infty$

(iii) *there exists an* α_0 *such that for every* $a > 0$,

(4.1) $$\int_0^a K(\alpha,x)dx \to 0 \quad \text{as } \alpha \to \alpha_0 ,$$

then

(4.2) $$\lim_{\alpha \to \alpha_0} T_\alpha f = 0,$$

where we define

(4.3) $$T_\alpha f = \int_0^\infty K(\alpha,x) f(x)dx .$$

The proof of theorem 46 results from the fact that it is easily seen to be true for step functions which form a dense subset of L_1-functions, and its validity for general L_1-functions follows upon application of theorem 45.

The kernel $K(\alpha,x) = e^{i\alpha x}$ in $0 < \alpha < \infty$ with $\alpha_0 = +\infty$ satisfies all the requirements, since

(4.4) $$\int_0^a e^{i\alpha x}\, dx = \frac{e^{i\alpha a}-1}{i\alpha} = O(\tfrac{1}{\alpha})$$

and (4.2) is then the Riemann Lebesgue Lemma.

Actually, we may also choose $K(\alpha,x) = e^{i\alpha x^q}$ for any $q > 0$, since in this case

$$\int_0^a K(\alpha,x)dx = \frac{1}{q} \int_0^{a^q} e^{i\alpha y}\, y^{\frac{1}{q}-1}\, dy ,$$

and for a fixed a, this tends to 0 as $\alpha \to \infty$ by the very same Riemann Lebesgue Lemma. More generally, we may choose

$$K(\alpha,x) = \exp\left(i\alpha \int_0^x \phi(P)dP\right)$$

where $\phi(P)$ is real-valued and Lebesgue integrable in every finite interval $0 \leq P \leq P_0$.

<u>THEOREM 47</u>. <u>Given two Banach spaces</u> B_1 <u>and</u> B_2, <u>and a bounded linear transformation</u> T <u>defined on a dense linear subset</u> $\{g\}$ <u>of</u> B_1 <u>such that</u> $Tg_1 \in B_2$, <u>then</u> T <u>can be extended uniquely to a bounded linear transformation on the whole space</u> B_1 <u>to</u> B_2.

Proof. For any $g \in \{g\}$ we have $\| Tg\| \leq M\| g\|$. If $f \in B_1$, there exists a sequence g_n in $\{g\}$ such that $\| g_n - f\| \to 0$. Since T is linear,

$$\| Tg_m - Tg_n\| = \| T(g_m - g_n)\| \leq M \| g_m - g_n\| \to 0$$

so that $\| Tg_m - Tg_n\| \to 0$, and since B_2 is complete, there exists a $\phi \in B_2$ such that $Tg_n \to \phi$. Also for given f, the element ϕ is independent of the sequence $\{g_n\}$ chosen. Now define $Tf = \phi$. The T so extended is obviously linear; it is bounded, because

$$\| Tf\| - \| Tg_n - Tf\| \leq \| Tg_n\| \leq \| Tf\| + \| Tg_n - Tf\|$$

or,

$$\left| \| Tg_n\| - \| Tf\| \right| \leq \| Tg_n - Tf\| \leq \varepsilon_n \to 0$$

and

$$| M \| g_n \| - M \| f_n \| | \leq M \| g_n - f \| \leq \delta_n \to 0$$

so that

$$\| Tf \| - \epsilon_n < \| Tg_n \| \leq M \| g_n \| < M \| f \| + \delta_n$$

and hence

$$\| Tf \| \leq M \| f \| .$$

The uniqueness of the extension follows from the fact that the limit in a B-space is unique, and from the fact that for T to be continuous it is necessary that $Tg_n \to Tf$ whenever $g_n \to f$.

<u>THEOREM 48</u>. <u>If the transformation</u> T <u>defined in theorem</u> 47 <u>is such that</u> $\| Tg \| = M \| g \|$ <u>for</u> $g \in \{g\}$, <u>then</u> (<u>for the extended</u> T) $\| Tf \| = M \| f \|$ <u>for all elements</u> $f \in B_1$.

Proof:

$$| \; \| Tg_n \| - \| Tf \| \; | \leq \| Tg_n - Tf \| \to 0$$

and

$$| \; \| g_n \| - \| f \| \; | \leq \| g_n - f \| \to 0$$

so that

$$\lim_{n\to\infty} \| Tg_n \| = \| Tf \| = \lim_{n\to\infty} M \| g_n \| = M \| f \| .$$

§5. L_p-spaces.

Take a finite or infinite interval $a \leq x \leq b$, and at first denote by $L_p(a,b)$ the class of Lebesgue measurable functions in (a,b) for which

$$\int_a^b |f(x)|^p dx < \infty \ , \quad 1 \leq p < \infty .$$

If $p > 1$, and if $\frac{1}{p} + \frac{1}{q} = 1$ then q is called the conjugate index of p, and $L_q(a,b)$ the conjugate class of $L_p(a,b)$.

We shall now prove two fundamental inequalities for functions in $L_p(a,b)$.

THEOREM 49. (Hölder's Inequality)

If $f(x) \in L_p(a,b)$, $p > 1$, *and* $g(x) \in L_q(a,b)$ *then* $f(x).g(x) \in L_1(a,b)$ *and*

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}} .$$

Proof: We may assume that $f \neq 0$ and $g \neq 0$. Consider the function

$$f(t) = t - \frac{t^p}{p} - \frac{1}{q}$$

for $t \geq 0$. We have $f(1) = f'(1) = 0$; furthermore, $f'(t) > 0$, for $0 < t < 1$, and $f'(t) < 0$ for $t > 1$. Hence $f(t) < 0$ for all $t \geq 0$ except for $t = 1$, when $f(t) = 0$. Thus

$$t \leq \frac{t^p}{p} + \frac{1}{q} \ , \quad t \geq 0 \ ,$$

(the equality holding only for $t = 1$). Setting $t = |A| . |B|^{1-q}$, and multiplying by B^q, we obtain

$$(5.1) \qquad |A.B| \leq \frac{|A|^p}{p} + \frac{|B|^q}{q} .$$

If we choose

$$A = \frac{f(x)}{(\int_a^b |f(x)|^p dx)^{\frac{1}{p}}} , \qquad B = \frac{g(x)}{(\int_a^b |g(x)|^q dx)^{\frac{1}{q}}}$$

then we have

$$\int_a^b |A|^p \, dx = \int_a^b |B|^q \, dx = 1 ;$$

now the product A.B is measurable (on account of a well - known property of measurable functions), and since the right side of (5.1) is integrable, it follows that |A.B| is integrable. Hence integrating (5.1) we get the required result

$$\left| \int_a^b A.B \, dx \right| \leq \int_a^b |A \, B| dx \leq \frac{1}{p} + \frac{1}{q} = 1 .$$

<u>THEOREM 50.</u> (<u>Minkowski's Inequality</u>)

<u>If both</u> f(x) <u>and</u> g(x) <u>belong to</u> $L_p(a,b)$, $p \geq 1$, <u>then we have</u>:

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}} .$$

Proof: We may suppose that $f(x) \geq 0$ and $g(x) \geq 0$. Now

$$\int_a^b |f+g|^p dx = \int_a^b |f+g|^{p-1}.f \, dx + \int_a^b |f+g|^{p-1} g \, dx .$$

Applying the inequality of the previous theorem, we obtain

$$\int_a^b |f+g|^p\,dx \leq \left(\int_a^b |f+g|^p\,dx\right)^{\frac{1}{q}} \cdot \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}}$$
$$+ \left(\int_a^b |f+g|^p\,dx\right)^{\frac{1}{q}} \cdot \left(\int_a^b |g|^p\,dx\right)^{\frac{1}{p}}$$

which is the result we require.

If we now introduce the quantity

$$\|f\| = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

then, by theorem 50 it has properties (2) and (3) of a norm (see §3), but not property (1), since the vanishing of

$$\int_a^b |f(x)|^p\,dx$$

is compatible with f(x) being $\neq 0$ on a set of measure zero (and only then). However property (1) of the norm can again be secured by a suitable concept of equivalence, this time any two functions f(x) being equivalent if differing only on a set of measure zero. The resulting normed linear space is a so-called L_p-space, and the first important fact about it is that it is also a Banach space according to the following theorem on Lebesgue integral, which we will quote without proof.

<u>THEOREM 51</u>. <u>If</u> $\{f_n(x)\}$ <u>is a sequence of functions each of which belongs to</u> $L_p(a,b)$, $p \geq 1$ <u>and if, for</u> $\varepsilon > 0$,

$$\int |f_j - f_k|^p\,dx < \varepsilon, \qquad j,k > j_0(\varepsilon),$$

then there exists a subsequence $f_{r_m}(x)$ *converging almost everywhere to some* $f(x)$ *in* $L_p(a,b)$, *and*

$$\left(\int |f_m - f|^p \, dx\right)^{\frac{1}{p}} \to 0 \quad \text{as } m \to \infty .$$

All the above properties of L_p-spaces hold in general types of measure spaces; in particular, in Euclidean spaces of k-dimensions, $k \geq 1$, and we can operate with any measurable set A instead of the interval (a,b). For a bounded interval (a,b) --- and more generally, for a set A of bounded measure --- a function g(x) of class $L_p(a,b)$, $p > 1$, is also an element of $L_1(a,b)$. In fact if we apply Hölder's inequality to the functions $|g(x)|$, 1, we obtain

$$\int_a^b |g(x)| dx \leq \left(\int_a^b |g(x)|^p \, dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b 1^q \, dx\right)^{\frac{1}{q}} = (b-a)^{\frac{1}{q}} \, \|g\|_p \; .$$

More generally, for $1 \leq p' < p$, any function of class $L_p(a,b)$ is also an element of class $L_{p'}(a,b)$. But this is no longer true, for instance, in the interval $(0,\infty)$, the Lebesgue measure of the interval being infinite.

Now, take a finite or infinite interval (a,b) (or, more generally a set A). A function f(x) in it is called finitely valued if there exists a finite number of disjoint measurable subsets $A_1, \ldots, A_n$ of (a,b) each of finite measure such that f(x) has a constant value c_k on A_k, $k = 1,\ldots,n$ and the value zero on the set which is complementary to $A_1 + \ldots + A_n$, in (a,b). Now it follows from general principles of Lebesgue measure that for each p, $1 \leq p < \infty$, the family of these functions is dense in

$L_p(a,b)$ in the given L_p norm. This is true for general spaces as well. However, in the Euclidean cases, more than that is true. There, if we deal with any interval (a,b), or in the k-dimensional case, if the basic set A is a box $\prod_{k=1}^{n}(a_k,b_k)$, then it suffices to take only those finitely-valued functions for which each of the sets $A_1, A_2, \ldots, A_n$ is itself a box I_k. In other words, the family of what we previously termed "step-functions" is already dense in $L_p(A)$ for every $p \geq 1$. Now step-functions are not continuous functions but only Baire functions of Baire class one. However, it can be seen that every such Baire function is L_p-approximable by a sequence of continuous functions. Therefore we obtain the useful result that the family of continuous functions is likewise dense in $L_p(E_k)$, which result can again be generalized to a general measure-space with a suitable topology in it. But in the Euclidean case one can go much further; there, for instance, the class of differentiable functions is already L_p-dense for every p, but this refinement will no longer be needed in the present context.

§6. Continuity, summability and approximation in L_p-norm

THEOREM 52. *If* $p \geq 1$ *and* $f(x) \in L_p(-\infty, \infty)$ *then*

$$\tau_f(h) = \left(\int_{-\infty}^{\infty} |f(x+h) - f(x)|^p \, dx\right)^{\frac{1}{p}} \to 0$$

as $h \to 0$; *more generally,* $\tau_f(h)$ *is continuous in* h.

Proof. Plainly, $\tau_f(h) \leq 2\|f\|$. Given $\varepsilon > 0$ there exists a continuous function ϕ vanishing outside a finite interval such that

$$\int_{-\infty}^{\infty} |f - \phi|^p \, dx < \varepsilon.$$

Set

$$\Phi(h) = \left(\int_{-\infty}^{\infty} |\phi(x+h)-\phi(x)|^p \, dx\right)^{\frac{1}{p}}$$

Then by Minkowski's inequality,

$$|\tau_f(h) - \Phi(h)| \leq \left(\int |f(x+h) - \phi(x+h)|^p \, dx\right)^{\frac{1}{p}}$$

$$+ \left(\int |f(x) - \phi(x)|^p \, dx\right)^{\frac{1}{p}}$$

$$< 2\varepsilon .$$

Since $\Phi(h)$ is continuous, we deduce that $\tau_f(h)$ is also continuous.

THEOREM 53. *If* $f(x) \in L_p(-\infty,\infty)$, $p \geq 1$, *and* $K(t) \in L_1(-\infty,\infty)$ *and if we set up*

$$\phi^K(x) = \int_{-\infty}^{\infty} f(x-t)K(t)dt$$

then the integral exists for almost all x, *and* $\phi^K(x) \in L_p(-\infty,\infty)$, *and* $\|\phi\| \leq M\|f\|$, *where*

$$M = \int_{-\infty}^{\infty} |K(t)|dt .$$

Proof: We have

$$|\phi^K(x)| \leq \int |f(x-t)K^{\frac{1}{p}}(t)|\,.\,|K(t)^{\frac{1}{q}}|dt$$

$$\leq \left(\int |f(x-t)|^p|K(t)|dt\right)^{\frac{1}{p}}\left(\int |K(t)|dt\right)^{\frac{1}{q}} .$$

Hence

$$|\phi^K(x)|^p \leq \left(\int |f(x-t)|^p|K(t)|dt\right)\left(\int |K(t)|dt\right)^{\frac{p}{q}}$$

Integrating this inequality and using Fubini's theorem, we obtain

$$\int_{-\infty}^{\infty} |\phi^K(x)|^p\, dx \leq \left(\int_{-\infty}^{\infty} |f(x-t)|^p\, dx\right)\left(\int_{-\infty}^{\infty} |K(t)|dt\right)^{1+\frac{p}{q}}$$

$$= \left(\int |f(x)|^p\, dx\right)\left(\int_{-\infty}^{\infty} |K(t)|dt\right)^p$$

hence

$$\| \phi^K(x)\|_p \leq \| f\|_p .\ M.$$

<u>Remarks</u>: Theorems 52 and 53 have obvious extensions to E_k.

We recall that for p = 1, we established the one-dimensional case (k = 1) of the formula:(7.6 Chr.I)

(6.1) $$S_R(x) = \frac{1}{2^{k-1}\Gamma(k/2)^{k/2}}\ R^k \int_{E_k} f(x+t)H(Rt)dV_t .$$

(In Chr.I we had S_R^K instead of S_R), where

(6.2) $$\frac{1}{(2\pi)^k}\int_{E_k} H(\alpha)dV_\alpha = 1 .$$

In the case of $f(x) \in L_p$, $p > 1$ we have not yet defined a Fourier transform, and so the expression $S_R(x)$ does not have the significance which it had in §7 of Chr.I. If we suppose that $H(\alpha) \in L_1$, then by theorem 53, it follows that $S_R(x) \in L_p$ (where $S_R(x)$ is now defined by (6.1)) whenever $f(x) \in L_p$, and further

$$\| S_R(x) \| < c \| f \|$$

where c is a constant (as usual). Also, if $t = (t_1, \ldots, t_k)$ then

$$\| S_R(x) - f(x) \|_p \leq c \int_{E_k} |f(x+\tfrac{t}{R}) - f(x)|^p H(t) dV_t$$

$$\leq c \int_{E_k} |\tau_f(t/R)|^p H(t) dV_t$$

$$\to 0$$

as $R \to \infty$, since $\tau_f(t/R)$ is a bounded continuous function tending to zero as $R \to \infty$, which is also continuous at the origin. Hence we have the following

THEOREM 54. *If* $f(x) \in L_p(E_k)$, *and* $H(\alpha)$ *satisfies the conditions*:

$$\frac{1}{(2\pi)^k} \int_{E_k} H(\alpha) dV_\alpha = 1, \quad H(\alpha) \in L_1,$$

then

$$\| S_R(x_1, \ldots, x_k) - f(x_1, \ldots, x_k) \|_p \to 0,$$

as $R \to \infty$, *where* S_R *is defined as in* (6.1).

Instead of approximation in norm considered in theorem 54, we may consider point-wise approximation of $S_R(x)$ to $f(x)$. We have

$$|S_R(x)-f(x)| \leq R^k\left[\int_{|t|<\delta}|f(x+t)-f(x)|H(Rt)dV_t + \int_{|t|\geq\delta}\right]$$

$$= I_1 + I_2, \quad \text{say},$$

where

$$|t|^2 = t_1{}^2 + \dots + t_k{}^2 .$$

Now,

$$|I_2| \leq 2R^k\left(\int_{|t|\geq\delta}|f(x+t)|^p dV_t\right)^{\frac{1}{p}}\left(\int_{|t|\geq\delta}|H(Rt_1,\dots,Rt_k)|^q dV_t\right)^{\frac{1}{q}}$$

If we now assume that $H(\alpha)$ is a radial function, then we obtain:

$$|I_2| \leq c\, R^k\left(\int_\delta^\infty |H(xR)|^q\, x^{k-1}\, dx\right)^{\frac{1}{q}}$$

$$= c\, R^k\left(\int_{\delta R}^\infty |H(y)|^q\, y^{k-1} R^{-k}\, dy\right)^{\frac{1}{q}}$$

$$= c\, R^{k(1-\frac{1}{q})}\left(\int_{\delta R}^\infty |H(y)|^q\, y^{k-1}\, dy\right)^{\frac{1}{q}}$$

$$= c\, R^{\frac{k}{p}}\left(\int_{\delta R}^\infty |H(y)|^q\, y^{k-1}\, dy\right)^{\frac{1}{q}} .$$

If we assume further that

$$H(|t|) = O\left(\frac{1}{|t|^{k+\varepsilon}}\right), \quad \varepsilon > 0$$

then it follows that

$$|I_2| = o(1), \quad \text{as } R \to \infty .$$

Again,

$$|I_1| = o(1)$$

on using the same argument as in theorems 6,7,35 and 36, provided that

$$x^k H(x) \to 0, \quad \text{as } x \to \infty$$

while

$$\int_0^t g_x(t)dt = o(t), \quad \text{as } t \to 0 ,$$

where $g_x(t)$ is defined as in §3 of Chapter II. We are thus in a position to state the following

THEOREM 55. If

(i) $f(x) \in L_p(E_k)$, $p > 1$

(ii) $H(\alpha)$ is a radial function, and $H(\alpha) \geq 0$,

(iii) $\frac{1}{(2\pi)^k} \int_{E_k} H(\alpha)dV_\alpha = 1$

(iv) $H(|\alpha|) = O(\frac{1}{|\alpha|^{k+\varepsilon}})$ as $|\alpha| \to \infty$, $\varepsilon > 0$,

then the condition

$$\int_0^t g_x(t)dt = o(t), \quad \text{as } t \to 0$$

implies

$$S_R(x) - f(x) = o(1), \quad \text{as } R \to \infty .$$

Remark: (ii) and (iii) together imply that $H(\alpha) \in L_1$ in E_k.

CHAPTER IV

FOURIER TRANSFORMS IN L_2.

§1. Transformations in Hilbert space

In this chapter we shall be concerned with Fourier transforms in L_2. Since, for an arbitrary function $f(x) \in L_2(-\infty, \infty)$ the integral defining the Fourier transform $\phi(\alpha)$ does not exist as an ordinary Lebesgue integral, the L_2-theory will have features not encountered in the L_1-theory. We need a few preliminary notions.

The bilinear expression

(1.1) $$(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

is called the inner-product of f and g. The inner product exists whenever $f, g \in L_2(-\infty, \infty)$ because of Hölder-Schwarz inequality. Also it has the following properties:

(1.2) $$(g,f) = (\overline{f,g})$$

(1.3) $$(f,f) = \|f\|^2$$

(1.4) $$(f,ag) = \bar{a}(f,g),$$ a being a number.

(1.5) $$(f, g_1 + g_2) = (f,g_1) + (f,g_2)$$

(1.6) $$|(f,g)| \leq \|f\| \cdot \|g\|$$

(1.7) (f,g) is a continuous function in both its arguments.

For,

$$(f,g) - (f_0,g_0) = (f-f_0,g) + (f_0,g-g_0)$$

$$\begin{aligned} |(f,g) - (f_0,g_0)| &\leq |(f-f_0,g)| + |(f_0,g-g_0)| \\ &\leq \|f-f_0\| . \|g\| + \|g-g_0\| . \|f_0\| \\ &\leq \|f-f_0\| . \|g_0\| + \|g-g_0\| . \|f_0\| \\ &\quad + \|f-f_0\| . \|g-g_0\| . \end{aligned}$$

Given $\varepsilon > 0$, there exists a δ such that

$$\delta(\|f_0\| + \|g_0\|) + \delta^2 < \varepsilon$$

and then

$$|(f,g) - (f_0,g_0)| < \varepsilon$$

for

$$\|f-f_0\| < \delta , \quad \text{and} \quad \|g-g_0\| < \delta .$$

If S_1 and S_2 are two subsets of $L_2(-\infty,\infty)$, a transformation T of S_1 into S_2 is called <u>isometric</u> if for every pair of elements in its domain, we have $(Tf, Tg) = (f,g)$. T is called <u>unitary</u> if its domain and range coincide with L_2, and if $(Tf,Tg) = (f,g)$ for $f,g \in L_2$.

§2. <u>Plancherel's theorem</u>

Our object in this section is to define a Fourier transform for functions in L_2, and to show that it represents a unitary transformation in L_2. In order to exhibit clearly the invertive properties of the Fourier transform in L_2, it is convenient to alter the constants occurring in our definitions in the L_1-theory. Hereafter we shall say that if $f(x) \in L_1(-\infty,\infty)$ then its Fourier transform $\phi(\alpha)$ is defined by:

$$\phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, dx . \tag{2.1}$$

With this definition in mind, we shall collect here two of the results we need from the L_1-theory. If S is the class of <u>bounded</u> functions belonging to $L_1(-\infty,\infty)$, then S is a dense subset of $L_2(-\infty,\infty)$, and if $f(x) \in S$, the Fourier transform is defined by the integral on the right side of (2.1), which exists for <u>every</u> α in $-\infty < \alpha < \infty$. We then have, first:

if $f, \phi \in S$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(y) e^{-ixy}\, dy \tag{2.2}$$

almost everywhere (see theorem 8).

Secondly, for $f, \phi \in S$ we have:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\phi(y)|^2\, dy \tag{2.3}$$

(See theorem 12). Finally, if $\phi^{\varepsilon}_{a,b}(y)$ is defined as follows, for $b > a$ and $\varepsilon > 0$:

$$\phi^{\varepsilon}_{a,b}(y) = \begin{cases} 0, & y < a-\varepsilon \\ 1, & a \leq y \leq b \\ \frac{y-a+\varepsilon}{\varepsilon}, & a-\varepsilon \leq y \leq a \\ \frac{b+\varepsilon-y}{\varepsilon}, & b \leq y \leq b+\varepsilon \\ 0, & y > b+\varepsilon \end{cases}$$

then the linear manifold D determined by functions of the form $\phi^{\varepsilon}_{a,b}$ is dense in L_2, because its closure contains, in particular, the step-functions.

Now, <u>define</u> $f^{\varepsilon}_{a,b}(x)$ by the equation:

$$f^{\varepsilon}_{a,b}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi^{\varepsilon}_{a,b}(y) e^{-iyx}\, dy. \tag{2.4}$$

Then, obviously, $f^{\varepsilon}_{a,b}(x)$ is a bounded function of class L_1, and in particular (by (2.2) and (2.3)) we have:

$$\phi^{\varepsilon}_{a,b}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{\varepsilon}_{a,b}(x) e^{ixy}\, dx \tag{2.5}$$

almost everywhere, and

$$\int_{-\infty}^{\infty} |\phi^{\varepsilon}_{a,b}(y)|^2\, dy = \int_{-\infty}^{\infty} |f^{\varepsilon}_{a,b}(x)|^2\, dx$$

Define a transformation Tf for $f \in S$ by the relation:

$$Tf = \phi, \quad f \in S \tag{2.6}$$

where ϕ is as in (2.1). Because of (2.3), T is an isometric transformation, and since S is dense in L_2, T can be abstractly extended to all $f \in L_2$, and the extension $\overline{T}$ is again isometric and unique, by theorems 47 and 48. (The extension $\overline{T}$ is, in fact, obtained by closing the operation

T given on S, and $\overline{T} = T$ on S). This defines:

$$(2.7) \qquad \phi(y) = \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ixy}\, dx, \qquad f \in L_2 .$$

"l.i.m." is to be read as "limit in mean". The precise meaning of (2.7) is as follows: given any $f \in L_2(-\infty, \infty)$ consider any sequence $\{f_n\}$, $f_n \in S$, such that $f_n \to f$ in L_2-norm. Set $\phi_n = Tf_n$; then $\phi \in L_2$ is the limit, in L_2-norm, of the sequence $\{\phi_n\}$, this limit existing, and we define $\overline{T}f = \phi$. It should be noted that $\{f_n\}$ may be <u>any</u> sequence in S converging towards f, and that ϕ is independent of the choice of $\{f_n\}$.

The class L_2 contains, in particular, the functions $\{f^{\xi}_{a,b}\}$ of the form described in (2.4); their images $\{\phi^{\xi}_{a,b}\}$ together with linear combinations constitute D, which is dense in L_2. Now, the range of $\overline{T}$ is a closed subset of L_2, and it contains D which is dense in L_2. Hence the range of $\overline{T}$ is the entire space L_2.

Being isometric, $\overline{T}f$ is one-one, and since the range of $\overline{T}$ covers the whole of L_2, there exists an inverse $\overline{T}^{-1}$, which is again isometric; in other words, $\overline{T}$ is unitary.

Just as in (2.6) and (2.7), if we <u>start</u> with $\psi^{\xi}_{a,b}(y) \in S$, the relation

$$g^{\xi}_{a,b}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^{\xi}_{a,b}(y)e^{-ixy}\, dy$$

gives rise to an isometric transformation T^*, say, in L_2, which carries ψ into g, where g is defined as follows:

$$g(x) = \text{l.i.m.}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(y) e^{-iyx}\, dy,\ \psi \in \mathbb{L}_2 \tag{2.8}$$

and l.i.m. has the same meaning as in (2.7). We shall now show that

$$T^* = \bar{T}^{-1} . \tag{2.9}$$

In fact, we have

$$\phi^{\xi}_{a,b}(y) = \bar{T} f^{\xi}_{a,b}(x) ,$$

and hence $f^{\xi}_{a,b}(x) = \bar{T}^{-1}\phi^{\xi}_{a,b}(y)$. However, by (2.4), $f^{\xi}_{a,b}(x) = T^*\phi^{\xi}_{a,b}(y)$. Hence

$$\bar{T}^{-1} = T^*$$

on a dense linear subset D of L_2; since both the operations are bounded, it follows that $\bar{T}^{-1} = T^*$ throughout L_2. Hence, if

$$\phi(y) = \text{l.i.m.}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ixy}\, dx \tag{2.10}$$

then

$$f(x) = \text{l.i.m.}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(y) e^{-iyx}\, dy . \tag{2.11}$$

Also, if $\phi(y) = \bar{T}f$, and $\psi(y) = \bar{T}g$ (as defined by (2.10) then, since $\bar{T}$ is isometric, we have the Parseval relation,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \phi(y)\overline{\psi(y)}dy . \tag{2.12}$$

In particular,

$$(2.13) \qquad \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\phi(y)|^2 \, dy \ .$$

From (2.10) and the theory of convergence in norm, it follows that

$$\overline{T}[f(x+h)] = e^{-ihy}\,\phi(y); \quad \overline{T}[\overline{f}(x)] = \overline{\phi(-y)};$$

$$\overline{T}^{-1}[f(x+h)] = e^{ihy}\phi(-y); \; \overline{T}^{-1}[\overline{f}(x)] = \overline{\phi(y)};$$

and hence from (2.12) we can deduce the following relations

$$(a) \int_{-\infty}^{\infty} f(x)g(-x)dx = \int_{-\infty}^{\infty} \phi(y)\psi(y)dy,$$

$$(b) \int_{-\infty}^{\infty} f(x)g(x)e^{-ihx}\, dx = \int_{-\infty}^{\infty} \phi(y)\psi(h-y)dy \ ,$$

$$(c) \int_{-\infty}^{\infty} f(x)g(h-x)dx = \int_{-\infty}^{\infty} \phi(y)\psi(y)e^{-ihy}\, dy,$$

(2.14)

all integrals being absolutely convergent. From (c) it follows that the Fourier transform of the convolution of two functions in L_2 is the product of their transforms.

Let $f \in L_2$ and $\overline{T}f = \phi$. Let $\psi_{a,b}(y) = 1$ for $a \le y \le b$, and $\psi_{a,b}(y) = 0$ for $y < a$ or $y > b$. Then

$$g_{a,b}(x) = \overline{T}^{-1}\psi_{a,b}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_{a,b}(y)e^{-iyx}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{-iyx}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \; \frac{e^{-ixb} - e^{-ixa}}{-ix}$$

Using (2.14) we get

$$\int_{-\infty}^{\infty}\phi(y)\psi_{a,b}(y)dy = \int_{-\infty}^{\infty}f(x)g_{a,b}(-x)dx \tag{2.15}$$

or

$$\int_a^b\phi(y)dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,\frac{e^{ibx}-e^{iax}}{ix}\,dx \tag{2.16}$$

for any $f \in L_2$. This may be considered as the <u>second interpretation</u> of relation (2.7).

If on the other hand, $f \in L_1 \cap L_2$, then from (2.15) we have

$$\int_a^b\phi(y)dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)dx\int_a^b e^{iyx}\,dy$$

and the integration on the right can now be interchanged, with the result

$$\int_a^b\phi(y)dy = \int_a^b dy\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iyx}\,f(x)dx.$$

Hence for almost all y,

$$Tf = \phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iyx}f(x)dx. \tag{2.17}$$

Next, if $f(x) \in L_2$, and we set

$$f_n(x) = \begin{cases} f(x), & \text{for } |x| \leq n \\ 0, & \text{for } |x| > n \end{cases}$$

then each $f_n(x) \in L_1 \cap L_2$, and $f_n(x) \to f(x)$ in L_2-norm. Hence, by (2.6) and (2.17) we have

$$\phi_n = Tf_n = \frac{1}{\sqrt{2\pi}}\int_{-n}^{n}f(x)e^{iyx}\,dx. \tag{2.18}$$

This is the _third interpretation_ of relation (2.7).

If in (2.16) we choose $a = o$ and $b = t$, we have:

$$\int_o^t \phi(y)dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{itx}-1}{ix} f(x)dx.$$

Hence for almost all t,

(2.19) $$\phi(t) = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} \frac{e^{itx}-1}{ix} f(x)dx$$

the derivative existing.

Inversely, for almost all x,

(2.20) $$f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} \frac{e^{-itx}-1}{-ix} \phi(x)dx,$$

the derivative existing.

(2.19) and (2.20) serve as _a fourth interpretation_ of relation (2.7).

We are now in a position to assert the following:

THEOREM 56. (Plancherel's theorem).

Let $f(x) \in L_2(-\infty, \infty)$. _Then the functions_

$$\phi_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x)e^{iyx}\, dx$$

converge in L_2_-norm to a function_ $\phi(y) \in L_2$, _and the transformation_ $\overline{T}$ _defined by_

$$\overline{T}f(x) = \phi(y) = \underset{n\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x)e^{iyx}\, dx$$

is a unitary transformation of L_2 _into itself_. _Similarly_, *the functions*

$$\psi_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} g(x)e^{-iyx}\, dx, \quad g(x) \in L_2,$$

converge in L_2-norm to a function $\psi(y) \in L_2$ and the transformation T^* defined by

$$T^* g = \psi(y) = \underset{n\to\infty}{\text{l.i.m}} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} g(x)e^{-iyx}\, dx$$

is the inverse of the transformation $\bar{T}$, so that whenever the relation

$$\phi(y) = \underset{n\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x)e^{iyx}\, dx$$

holds, then so does the other:

$$f(x) = \underset{n\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \phi(y)e^{-iyx}\, dy$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\phi(y)|^2\, dy \; .$$

§3. General summability

Let $f(x) \in L_2$, $K(t) \in L_2$ and $\bar{T}f = \phi(y)$, $\bar{T}K = H(y)$, where K,H are real. Let $H(y) \in L_1 \cap L_2$. Set

$$S_R^K(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(y)K(y/R)e^{-iyx}\, dy \; . \tag{3.1}$$

Using Parseval's relation we have

$$S_R^K(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+y)RH(Ry)dy \tag{3.2}$$

for every fixwd x (and R). Also $S^K_R(x) \in L_2$ (see theorem 53). Hence, as in theorem 54, we have:

<u>THEOREM 57</u>. <u>If</u> $f(x) \in L_2$ <u>and</u> $K(t) \in L_2$, <u>and</u> $\overline{T}K(t) = H(y) \in L_1 \cap L_2$, <u>then</u>

(3.3) $$\lim_{R\to\infty} \| S^K_R(x) - f(x) \| = 0 ,$$

<u>where</u> $S^K_R(x)$ <u>is defined by</u> (3.1).

Again, using theorem 55, we obtain:

<u>THEOREM 58</u>. <u>If</u> $f(x) \in L_2$ <u>and</u> $K(t) \in L_2$, <u>and</u> $H(y) = O(\frac{1}{y^{1+\varepsilon}})$, $\varepsilon > 0$, <u>as</u> $y \to \infty$ <u>where</u> $H(y) = \overline{T}K(t)$, <u>and, further</u>

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} H(\alpha)d\alpha = 1$$

<u>then</u>

(3.4) $$S^K_R(x) \to f(x) \quad \text{as } R \to \infty$$

<u>for almost all</u> x, <u>provided that</u>

(3.5) $$\frac{1}{t} \int_0^t g_x(t)dt \to 0 \quad \text{as } t \to 0$$

<u>where</u>

$$g_x(t) = \frac{f(x+t)+f(x-t)}{2} - f(x).$$

Dually, we also have, in like manner, for $\phi \in L_2$,

(3.6) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)K(\tfrac{x}{R})e^{iyx}\, dx \to \phi(y), \quad \text{as } R \to \infty ,$$

for almost all y, provided that

(3.7) $$\phi_y(t) = \frac{\phi(y+t)+\phi(y-t)}{2} - \phi(y)$$

is such that

$$\frac{1}{t}\int_0^t \phi_y(t)dt \to 0 \quad \text{as } t \to 0.$$

Remarks: If $|H(y)| \leq H_0(y)$, where $H_0(y)$ now satisfies the condition that H satisfied in theorem 58, then

$$\frac{1}{t}\int_0^t |g_x(s)|ds = o(1) \quad \text{as } t \to 0$$

will imply (3.4), while

$$\frac{1}{t}\int_0^t |\phi_y(s)|ds = o(1) \quad \text{as } t \to 0$$

will imply (3.6).

If

$$K(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

then

$$H(y) = \left(\frac{\sin y/2}{y/2}\right)^2$$

and

$$H_0(y) = \frac{1}{1+y^2}$$

and hence we have

(3.8) $$\lim_{R\to\infty} \frac{1}{R} \int_0^R \phi_\rho(y)d\rho = \phi(y)$$

almost everywhere, where

$$\phi_\rho(y) = \int_{-\rho}^{\rho} f(x)e^{iyx}\,dx$$

(3.8) may be looked upon as <u>a fifth interpretation</u> of (2.7).

In (3.6) we proved that if $f \in L_2$ and $\bar{T}f = \phi$, then the existence of

$$\text{(3.9)} \qquad \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)K\left(\frac{x}{R}\right)e^{iyx}\,dx$$

implies that it is equal to $\phi(y)$ almost everywhere. More generally we can state the following:

If

(i) $K_n(x) \in L_2$, for each n,

(ii) $|K_n(x)| \leq M$,

(iii) $\lim\limits_{n\to\infty} K_n(x) = 1$, for almost all x,

(iv) $f(x) \in L_2$, and $\bar{T}f = \phi$

(v) $S_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)K_n(x)e^{ixy}\,dx$

and $\lim\limits_{n\to\infty} S_n(y)$ exists for a measurable set in y, say B_y, then

$$\text{(3.10)} \qquad \lim_{n\to\infty} S_n(y) = \phi(y), \qquad y \in B_y .$$

For, if $f_n(x) = f(x)K_n(x)$, then $f_n(x) \to f(x)$ in L_2-norm. And $S_n(y) = \bar{T}f_n(x)$, where $\bar{T}$ is continuous. Hence $S_n(y) \to \phi(y)$ in L_2-norm. That is, $S_{n_k}(y) \to \phi(y)$ almost everywhere. In particular, on the measurable set B_y, $S_n(y)$ converges. Hence

$$\lim_{n\to\infty} S_n(y) = \phi(y), \quad y \in B_y .$$

Remark: Note that (3.10) does not assert that $\lim_{n\to\infty} S_n(y)$ exists, but only says that, if the limit exists then it is equal to the transform.

§4. Several variables.

The extension of Plancherel's theorem from one to several variables does not involve any new difficulty in principle.

We start with the function $\phi^{\varepsilon}_{a,b}$ defined by:

$$(4.1) \qquad \phi^{\varepsilon}_{a,b}(y_1,\ldots,y_k) = \prod_{j=1}^{k} \phi^{\varepsilon}_{a_j,b_j}(y_j),$$

where each factor $\phi^{\varepsilon}_{a_j,b_j}(y_j)$ is exactly of the form described in §2. Then $\phi^{\varepsilon}_{a,b}(y_1,\ldots,y_k)$ is a bounded function belonging both to L_1 and L_2, and if we therefore define

$$(4.2) \quad f^{\varepsilon}_{a,b}(x_1,\ldots,x_k) = \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \phi^{\varepsilon}_{a,b} e^{-i\sum y_j x_j} dV_y$$

then, obviously, $f^{\varepsilon}_{a,b}$ is again a bounded function belonging to L_1 and L_2, and by theorems 38 and 39 we have,

$$\phi^{\varepsilon}_{a,b}(y_1,\ldots,y_k) = \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f^{\varepsilon}_{a,b}(x_1,\ldots,x_k) e^{i\sum x_j y_j} dV_x$$

(4.3)

everywhere. Furthermore

$$(4.4)\quad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\phi^{\xi}_{a,b}(y_1,\ldots,y_k)|^2 dV_y = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f^{\xi}_{a,b}(x_1,\ldots,x_k)|^2 dV_x .$$

Hence, if we define a transformation Tf for $f \in S$ (where S is the class of bounded functions belonging to L_1 and L_2, in E_k) by the relation

$$Tf = \phi , \quad f \in S$$

where

$$\phi(y_1,\ldots,y_k) = \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\ldots,x_k) e^{i\sum y_k x_k} dV_x$$

then Tf is an isometric transformation, and since S is dense in $L_2(E_k)$, T can be abstractly extended to all $f \in L_2(E_k)$, and the extension is again isometric and unique by theorems 47 and 48. This defines

$$(4.5)\quad \phi(y) = \text{l.i.m.} \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x) e^{i\sum x_j y_j} dV_x ,$$

where l.i.m. means that, if $f_n(x) \to f(x)$ in L_2-norm in E_k then ϕ is defined as the limit of $\phi_n(x)$, this limit existing, in L_2-norm.

It then follows as in the case of one variable that $\overline{T}$ is unitary and the inverse of T is defined by

$$\overline{T}^{-1}\phi = f ,$$

where

$$f(x_1,\ldots,x_k) = \text{l.i.m.} \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \phi \, e^{-i\sum x_j y_j} dV_y .$$

Being unitary, Parseval's relation also follows:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1,\ldots,x_k)\,\overline{g(x_1,\ldots,x_k)}\,dV_x \tag{4.6}$$
$$= \int_{-\infty}^{\infty} \phi(y_1,\ldots,y_k)\,\overline{\psi(y_1,\ldots,y_k)}\,dV_y$$

If A is a bounded measurable set in (E_k) and $w(y)$ is the characteristic function of A, then $w(y) \in L_2$; let $\overline{T}^{-1}w(y) = \psi(x)$. Thus for every $f \in L_2$, it follows from (4.6) that

$$\int_{E_k} w(y)\phi(y)\,dV_y = \int_{E_k} \psi(x) f(x)\,dV_x ,$$

or,

$$\int_A \phi(y)\,dV_y = \int_{E_k} f(x)\,dV_x \left(\int_A e^{i\sum y_j x_j} dV_y\right) .$$

If $f(x) \in L_1 \cap L_2$, then, since A is bounded, we have

$$\int_A \phi(y)\,dV_y = \int_A dV_y \left[\int_{E_k} e^{i\sum y_j x_j} f(x)\,dV_x\right] ;$$

this being true for every bounded measurable set A, we hence have

$$\phi(y) = \int_{E_k} e^{i\sum y_j x_j} f(x)\,dV_x \tag{4.7}$$

<u>almost everywhere</u> for $f \in L_1 \cap L_2$.

If $\{A_n\}$ is any sequence of bounded measurable sets such that

$$0 \subset A_1 \subset A_2 \subset \ldots \subset A_n \to E_k,$$

and if $f(x) \in L_2$, and we <u>define</u>:

$$f_n(x) = \begin{cases} f(x), & x \in A_n \\ 0 , & x \notin A_n \end{cases}$$

then, due to the boundedness of A_n, each $f_n(x) \in L_1 \cap L_2$, and $f_n(x) \to f(x)$ in L_2-norm. Hence, by (4.7) we have

$$(4.8) \quad \phi_n(y) = Tf_n(x) = \frac{1}{(2\pi)^{k/2}} \int_{A_n} f(x) e^{i\sum x_j y_j} dV_x$$

almost everywhere, and

$$(4.9) \quad \phi(y) = \underset{n\to\infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{k/2}} \int_{A_n} f(x) e^{i\sum x_j y_j} dV_x .$$

In particular, if A_n is the set defined by the property: $x \in A_n : \supset : |x|^2 \leq n^2$; $|x|^2 = x_1{}^2 + \ldots + x_k{}^2$, then we obtain

$$\phi(y) = \underset{n\to\infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{k/2}} \int \cdots \int_{x_1{}^2+\ldots+x_k{}^2 \leq n^2} f(x) e^{i\sum x_j y_j} dV_x .$$

Thus we have the extension of theorem 56:

<u>THEOREM 59</u>. <u>Let</u> $f(x_1,\ldots,x_k) \in L_2(E_k)$. <u>Then the functions</u>

$$\phi_n(y_1,\ldots,y_k) = \frac{1}{(2\pi)^{k/2}} \int \cdots \int_{\sum x_j{}^2 \leq n^2} f(x_1,\ldots,x_k) e^{i\sum x_j y_j} dV_x$$

<u>converge in</u> L_2<u>-norm to a function</u> $\phi(y_1,\ldots,y_k) \in L_2$ <u>and the transformation</u> $\bar{T}$ <u>defined by</u>:

$$\bar{T}f(x) = \phi(y) = \underset{n\to\infty}{\mathrm{l.i.m.}} \int \cdots \int_{\sum x_j{}^2 \leq n^2} f(x) e^{i\sum y_j x_j} dV_x$$

<u>is a unitary transformation of</u> $L_2(E_k)$ <u>into itself. Similarly, the functions</u>

$$\psi_n(y_1,\ldots,y_k) = \frac{1}{(2\pi)^{k/2}} \int \cdots \int_{\sum x_j{}^2 \leq n^2} g(x) e^{-i\sum y_j x_j} dV_x , g \in L_2(E_k)$$

converge in L_2-norm to a function $\psi(y) \in L_2(E_k)$ and the transformation T^* defined by

$$T^* g = \psi(y) = \underset{n\to\infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{k/2}} \int\cdots\int_{\sum x_j^2 \leq n^2} g(x) e^{-i\sum y_j x_j} dV_x$$

is the inverse of the transformation $\overline{T}$, so that whenever the relation

$$\phi(y) = \underset{n\to\infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{k/2}} \int\cdots\int_{\sum x_j^2 \leq n^2} f(x) e^{i\sum x_j y_j} dV_x$$

holds, then so does the other:

$$f(x) = \underset{n\to\infty}{\mathrm{l.i.m.}} \frac{1}{(2\pi)^{k/2}} \int\cdots\int_{\sum x_j^2 \leq n^2} \phi(y) e^{-i\sum y_j x_j} dV_y$$

and

$$\int_{E_k} |f(x)|^2 dV_x = \int_{E_k} |\phi(y)|^2 dV_y .$$

§5. Radial functions

Assume that $f(x_1,\ldots,x_k) \in L_2$ is radial in the sense defined in Chr.II; then

$$f(x_1,\ldots,x_k) = f(|x|) = f(\sqrt{x_1^2 + \ldots + x_k^2}).$$

Set

$$f_n(|x|) = \begin{cases} f(|x|), & \text{for } |x| < n \\ 0\ , & \text{for } |x| \geq n. \end{cases}$$

Then

$$f_n(x_1,\ldots,x_k) \to f(x_1,\ldots,x_k)$$

in L_2-norm, where $f_n \in L_1 \cap L_2$. Since $f_n \in L_1$, it follows that $\phi_n(y)$ is also radial, and by Plancherel's theorem, $\phi_n(y) \to \phi(y) = \overline{T}f$. Since the radial functions in L_2 form a closed subspace of L_2, it follows that $\phi(y)$ is also radial (and similarly for the inverse). Thus,

(5.1) if $f \in L_2$ is radial, then the Fourier transform of f is also radial.

Now, if f is radial, then $f \in L_2(E_k)$ if and only if $\tilde{f}(t)t^{k-1} \in L_2(0,\infty)$, where $\tilde{f}(t) = f(\sqrt{x_1^2+\ldots+x_k^2})$. Set

$$F(t) = \tilde{f}(t)t^{k-1} ,$$

and

$$\Phi(t) = \tilde{\phi}(t)t^{k-1} .$$

As before, we then have for $F(x) \in L_2(0,\infty)$,

$$\Phi(y) = \text{l.i.m} \int_0^\infty F(x)S_\mu(yx)dx ,$$

where $\Phi(y) \in L_2(0,\infty)$, and

$$F(x) = \text{l.i.m.} \int_0^\infty \Phi(y)S_\mu(yx)dy$$

and

$$\int_0^\infty F\,\overline{G}\,dx = \int_0^\infty \Phi\,\overline{\Psi}\,dy$$

where

$$\mu = \frac{k-2}{2}$$

Thus the inversion formulas in k-variables, if applied to radial functions, produce the so-called Hankel inversion formulas for $\mu = \frac{k-2}{2}$, and we also obtain their validity in L_2-norm. This manner of derivation, gives then only for $\mu =$ an integer or $\mu =$ half-integer. However, the formulas and their validity remain in force for arbitrary real values of μ, as we will show in the next chapter.

If k=1 the Hankel transform reduces to the cosine transform, while k=3 gives something very akin to the sine-transform.

If k=1, a radial function is just an even function, and hence as in (2.19)

$$\phi(y) = \frac{1}{\pi}\frac{d}{dy}\int_0^\infty f(x)\frac{\sin yx}{x}\,dx$$

almost everywhere. The right side is the derivative of an integral of an L_1 function, so $\phi(y)$ exists almost everywhere. For the inverse, we have:

$$f(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^\infty \phi(y)\frac{\sin yx}{y}\,dy$$

almost everywhere.

§6. Derivatives and their transforms

We will now have some operational theorems involving Fourier transforms in L_2. We will deal exclusively with the one-dimensional situation, although the k-dimensional analogues are frequently of considerable interest and far from trivial.

Some of the theorems to follow are extensions (remaining one-dimensional) to L_2-functions of theorems previously stated and proved for L_1-functions. In some such instances, the L_1-case is the more difficult one, and it often requires several elaborate manipulations, where the L_2 case could be disposed of in fewer steps, the reason being that in the L_2-case, norm-convergence of the function can be immediately translated into norm convergence of the transform (and conversely), whereas in the L_1-case no such translation is directly available but must be laboriously substituted for by other rules of procedure.

<u>THEOREM 60</u>. <u>If</u> $f_1(x) \in L_1$ <u>in</u> E_1, <u>and</u> $T[f_1(x)] = \phi_1(\alpha)$, <u>and</u> $f_2(x) \in L_2$ <u>in</u> E_1, <u>and</u> $\overline{T}[f_2(x)] = \phi_2(\alpha)$, <u>and if</u> $\phi_1(\alpha) = \phi_2(\alpha)$ <u>almost everywhere</u>, <u>then</u> $f_1(x) = f_2(x)$ <u>almost everywhere</u>.

Proof: Except for a null-set, we have, by theorem 6,

$$(6.1) \qquad f_1(x) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_1(\alpha) e^{-i\alpha x} e^{-\alpha^2/R^2} d\alpha = S_R^1 ,$$

and

$$(6.2) \qquad f_2(x) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_2(\alpha) e^{-i\alpha x} e^{-\alpha^2/R^2} d\alpha = S_R^2 ,$$

by theorem 58.

The right sides of (6.1) and (6.2) are equal, hence the left sides are equal almost everywhere.

<u>THEOREM 61</u>. <u>If</u> $H(y) \in L_2(E_1)$ <u>and</u> $G(x) = \overline{T}[H(y)]$, if $H(y)$ is absolutely continuous, and $H'(y) \in L_2(E_1)$,

then $\bar{T}[H'(y)] = -ix\ G(x)$ (which automatically belongs to $L_2(E_1)$).

Proof: We have

$$H'(y) = \text{l.i.m.}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} A(x)dx ,$$

where $A(x) = \bar{T}[H'(y)]$. Integrating this between finite limits, we get (see 2.20)

$$H(y+h) - H(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} \left(\frac{e^{-ihx}-1}{-ix}\right) A(x)dx .$$

On the other hand, we have

$$H(y+h)-H(y) = \text{l.i.m.}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx}(e^{-ihx}-1)G(x)dx .$$

Now

$$\frac{e^{-ihx}-1}{-ix} \in L_2(-\infty,\infty)$$

and

$$A(x) \in L_2(-\infty,\infty)$$

so that

$$\left(\frac{e^{-ihx}-1}{-ix}\right) A(x) \in L_1(-\infty,\infty)$$

while

$$(e^{-ihx}-1)G(x) \in L_2 .$$

Using theorem 60, we have

$$\left(\frac{e^{-ihx}-1}{-ix}\right) A(x) = (e^{-ihx}-1)\ G(x)$$

almost everywhere.

Take $h=1$ and $x \neq 2\pi n$; then

$$A(x) = (-ix)G(x)$$

almost everywhere. This is the L_2-analogue of th.3(ii).

<u>THEOREM 62</u>. <u>If</u> $f(x) \in L_2(E_1)$ <u>and</u> $ixf(x) \in L_2(E_1)$, <u>and</u> $\bar{T}[f] = \phi(\alpha)$, <u>then</u> $\phi'(\alpha)$ <u>exists and</u> $\bar{T}[ixf] = \phi'(\alpha)$.

Proof: Let

$$\phi_n(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x)e^{i\alpha x}dx$$

and

$$\psi_n(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} ixf(x)e^{i\alpha x}dx \ .$$

Then

$$\phi_n(\alpha) \to \phi(\alpha) \in L_2, \quad \text{and} \quad \psi_n(\alpha) \to \psi(\alpha) \in L_2.$$

However,

$$\psi_n(\alpha) = \phi_n'(\alpha) \ ,$$

the latter derivative obviously existing, and theorem 62 follows now from the

<u>LEMMA 4</u>. If on a finite or infinite α-interval we are given certain L_2-functions, and if the limit-relations

$$\phi_n(\alpha) \to \phi(\alpha) \tag{6.3}$$

$$\psi_n(\alpha) \to \psi(\alpha) \tag{6.4}$$

hold in L_2-norm, and if the functions $\psi_n(\alpha)$ are(locally) absolutely continuous, and

$$\psi_n(\alpha) = \phi_n'(\alpha)$$

then we also have

$$\psi(\alpha) = \phi'(\alpha).$$

Proof: If α and β are any(finite)points of the interval, then relations (6.3) and (6.4), first of all, imply

$$(6.5)\lim_{n\to\infty} \int_\alpha^\beta \phi_n'(x)dx = \lim_{n\to\infty} \int_\alpha^\beta \psi_n(\alpha)d\alpha = \int_\alpha^\beta \psi(\alpha)d\alpha ,$$

that is

$$(6.6) \qquad \lim_{n\to\infty} [\phi_n(\beta) - \phi_n(\alpha)] = \int_\alpha^\beta \psi(y)dy .$$

Since $\phi_n(x)$ converges in L_2-norm to $\phi(x)$, there exists a subsequence ϕ_{n_k} such that $\lim_{k\to\infty} \phi_{n_k}(x) = \phi(x)$ for almost all x. Let y be a value for which

$$\lim_{k\to\infty} \phi_{n_k}(y) = \phi(y).$$

Then, since

$$\lim_{k\to\infty} [\phi_{n_k}(\beta) - \phi_{n_k}(y)] = \int_y^\beta \psi(x)dx,$$

we have

$$\lim_{k\to\infty} \phi_{n_k}(\beta) = \phi(y) + \int_y^\beta \psi(x)dx .$$

Therefore ,

$$\phi(\beta) = \phi(y) + \int_y^\beta \psi(x)dx$$

for every β ; that is,

$$\psi(x) = \phi'(x)$$

as claimed.

Remarks: Theorem 62 is the L_2-analogue of theorem 3(i).

We say that a function f(x) is locally absolutely continuous, if it is absolutely continuous in every finite x-interval.

THEOREM 63. *If* $f(x) \in L_2(E_1)$, $g(x) \in L_2(E_1)$ *and* $\overline{T}f = \phi$ *and* $\overline{T}g = \psi = -i\alpha\,\phi(\alpha)$, *then* $g(x)$ *is locally absolutely continuous and*

(6.7) $$g(x) = f'(x);$$

also,

(6.8) $$f(x) = -\int_x^{\infty} g(y)dy$$

the integral existing conditionally.

Proof: By Plancherel's theorem, $\phi \in L_2$ and $-i\alpha\,\phi(\alpha) \in L_2$; hence (6.7) follows from theorem 62 and for any a,b we have

(6.9) $$f(y)-f(x) = \int_x^y g(z)dz .$$

However, by Schwarz's inequality we have

$$|f(y)-f(x)|^2 \leq \int_x^y |g(z)|^2 dz . \int_x^y dz$$
$$\leq (y-x) \int_{-\infty}^{\infty} |g(z)|^2 \, dz \; ;$$

therefore, $f(x)$ is uniformly continuous in $(-\infty, \infty)$. But if a uniformly continuous function $f(x)$ belongs to L_2 (or L_p) then it tends to zero as $x \to \infty$. Now letting $y \to \infty$ in (6.9) we obtain (6.8).

Theorem 63 is the L_2-analogue of theorem 15.

THEOREM 64. *If* $f(x) \in L_2(E_1)$, $g(x) \in L_2(E_1)$ *and* $\overline{T}f = \phi$, $\overline{T}g = \psi$, *and if* $\psi(\alpha) = (-i\alpha)^n \phi(\alpha)$, *where* n *is a positive integer, then* f *has* n *derivatives all belonging to class* L_2. *Hence* $(-i\alpha)^k \phi(\alpha)$, $k = 0,\ldots,n$ *are all transforms*.

Proof: If $\phi(\alpha) \in L_2$ and $(-i\alpha)^n\phi(\alpha) \in L_2$ for $n \geq 2$, then $(-i\alpha)\phi(\alpha) \in L_2$. Using this, we get the result by applying theorems 62 and 63. This is the L_2-analogue of theorem 16.

<u>THEOREM 65</u>. <u>If</u> $f(x) \in L_2(E_1)$ <u>and</u> $f(x)$ <u>has</u> n <u>derivatives and</u> $f^{(n)}(x) \in L_2(E_1)$, <u>then</u> $f^{(k)}(x) \in L_2(E_1)$ <u>for</u> $0 < k < n$.

Proof: If $\phi(\alpha) = \overline{T}f$ and $\psi(\alpha) = \overline{T}f^{(n)}$, put

$$S_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(y) e^{-y^2} e^{-iyx} dy; \tag{6.10}$$

then by (3.1) and (3.2) we have

$$S_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+t) H(t) dt \tag{6.11}$$

where $\overline{T}[e^{-x^2}] = H(\alpha)$.

By successive differentiation of (6.10) we get

$$D^n S_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-iy)^n \phi(y) e^{-y^2} e^{-iyx} dy, \tag{6.12}$$

the differentiation being justified by the uniform convergence of the derived integral in every finite x-interval. Since

$$f^{(k)}(x) = O(|x|^n), \quad \text{as } x \to \infty, \ 0 \leq k < n,$$

and $H(t) = e^{-t^2/4}$, we can differentiate (6.11) as well, and making use of the fact that $H(t)$ is even, we obtain

$$D^n S_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(n)}(x-y) H(y) dy. \tag{6.13}$$

Since $f^{(n)} \in L_2$ and $H(y) \in L_1$, by theorem 53, $D^n S_1(x) \in L_2$.

By (2.14) we have

$$(6.14) \quad D^n S_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} \psi(y) e^{-ixy} dy .$$

The functions $\psi(y)e^{-y^2}$ and $(-iy)^n \phi(y)e^{-y^2}$ both belong to $L_1(E_1)$. Hence by the uniqueness of Fourier transforms in L_1 and continuity of the transforms (in y), we conclude that

$$(-iy)^n \phi(y)e^{-y^2} = \psi(y)e^{-y^2}$$

which implies that

$$\psi(y) = (-iy)^n \phi(y) ,$$

and we now use theorem 64 and we obtain theorem 65.

<u>THEOREM 66</u>. <u>If</u> $f(x)$ <u>is locally absolutely continuous and</u> $f'(x) = g(x) \in L_2$, <u>then</u>

$$g_h(x) \to g(x)$$

<u>in</u> L_2-<u>norm</u> as $h \to 0$, <u>where</u>

$$g_h(x) = \frac{f(x+h)-f(x)}{h} .$$

Proof: Since

$$\frac{f(x+h)-f(x)}{h} - g(x) = \frac{1}{h} \int_0^h [g(x+y)-g(x)]dy$$

it follows that

$$\left|\frac{f(x+h)-f(x)}{h} - g(x)\right|^2 \le \frac{1}{h^2} \int_0^h |g(x+y)-g(x)|^2 dy \int_0^h dy .$$

Therefore

$$\int_{-\infty}^{\infty} |g_h(x)-g(x)|^2 dx \leq \frac{1}{h}\int_0^h dy \int_{-\infty}^{\infty} |g(x+y)-g(x)|^2 dy$$

$$= \frac{1}{h}\int_0^h \tau_g^2(y)dy$$

$$= o(1), \quad \text{as } h \to 0 ,$$

because $\tau_g^2(y) \to 0$ as $y \to 0$, by theorem 52.

<u>THEOREM 67</u>. <u>If</u> $f(x) \in L_2(E_1)$ <u>and if its difference - quotient</u> $g_h(x)$ <u>converges in</u> L_2<u>-norm to a function</u> $g(x)$ <u>as</u> $h \to 0$, <u>then</u> $f(x)$ <u>is locally absolutely continuous and</u> $g(x)$ <u>its derivative almost everywhere.</u>

Proof. Being an L_2-limit of L_2-functions $g(x) \in L_2$. Let $\phi(\alpha)$ and $\psi(\alpha)$ be the Fourier transforms of f and g respectively; then it follows that

$$g_h(x) = \underset{h\to 0}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\alpha} \frac{e^{-i\alpha h}-1}{h} \phi(\alpha)d\alpha .$$

Being L_2-convergent, the family $\{g_h(x)\}$ is also bounded in L_2-norm; hence, by Plancharel's theorem

$$\int_{-\infty}^{\infty} \left| \frac{e^{-i\alpha h}-1}{h} \right|^2 . |\phi(\alpha)|^2 d\alpha < M.$$

In particular,

$$\int_{-A}^{A} \left| \frac{e^{-i\alpha h}-1}{h} \right|^2 . |\phi(\alpha)|^2 d\alpha < M.$$

For fixed A, passing to the limit as $h \to 0$, we obtain

$$\int_{-A}^{A} \alpha^2 |\phi(\alpha)|^2 d\alpha < M$$

independently of A. Now letting $A \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \alpha^2 |\phi(\alpha)|^2 \, d\alpha < M.$$

Thus

$$-i\alpha\, \phi(\alpha) \in L_2(-\infty,\infty). \tag{6.15}$$

Since $g_h(x) \to g(x)$ in L_2-norm, we also have

$$\lim_{h \to 0} \int_{-\infty}^{\infty} \left| \frac{e^{-i\alpha h}-1}{h} \phi(\alpha) - \psi(\alpha) \right|^2 d\alpha = 0. \tag{6.16}$$

But, for variable h the integrand is majorized by $\alpha^2|\phi(\alpha)|^2 + |\psi(\alpha)|^2$ which belongs to L_2. Hence, we can let $h \to 0$ under the integral sign in (6.16), and this gives

$$\int_{-\infty}^{\infty} |-i\alpha\, \phi(\alpha) - \psi(\alpha)|^2 d\alpha = 0 .$$

Hence

$$\psi(\alpha) = -i\alpha\, \phi(\alpha)$$

almost everywhere; therefore, by theorem 63,

$$g(x) = f'(x)$$

almost everywhere.

§7. Boundary values

We are now going to transcribe theorems 23 and 24 of §15, Chr.I, from L_1-into L_2-functions. The L_2 approach is the easier one, and is much more familiar from the operational view-point. It is also the more important approach, in as much as, with the present-day methods, the L_2 procedure is available for differential equations and for problems of even greater complexity. But it would be a moot question to discuss whether it is also intrinsically the most natural one. In the task of translating partial differential equations into precisely given operational equations, sometimes the mathematician influences the physicist, but sometimes is also influenced by the physicist's preference, and there is as yet no firm guidance for deciding what, in a given case, would be mathematically the most natural approach.

Let $f(x) \in L_2$, $\bar{T}f = \phi(\alpha)$, and let

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\alpha) e^{-ix\alpha - y\alpha^2} d\alpha$$

(7*) $$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) \frac{1}{\sqrt{y}} e^{-(x-z)^2/4y} dz$$

$f(x,y)$ is obtained from $S_R^K(x)$ in (3.2) if we put $K(x)=e^{-x^2}$ and $y = \frac{1}{R^2}$.

<u>Properties of f(x,y)</u>:

(7.1) For every $y > 0$, $f(x,y) \in L_2$ as a function in x.

[In fact, its transform $\phi(y)e^{-y\alpha^2}$ belongs to L_2, if $\phi(\alpha)$ does.]

(7.2) $\lim_{y \to 0} f(x,y) = f(x)$ in L_2-norm

[In fact, $\phi(\alpha)e^{-y\alpha^2}$ converges in L_2-norm to $\phi(\alpha)$].

(7.3) For every $y > 0$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and belong to L_2.

In fact, if $f(x) \in L_2$ and $K(x) \in L_1$, then

$$f^K(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy$$

belongs to L_2, by theorem 53.

(7.4) For every $y > 0$, there exists a function $\frac{\partial f}{\partial y} \in L_2$ such that

$$\lim_{h\to 0} \left(\int_{-\infty}^{\infty} \left| \frac{f(x,y+h)-f(x,y)}{h} - \frac{\partial f}{\partial y} \right|^2 dx\right)^{\frac{1}{2}} = 0 .$$

In fact, the transform of $\frac{f(x,y+h)-f(x,y)}{h}$ is

(7.4a) $$\phi(\alpha)e^{-y\alpha^2} \left\{ \frac{e^{-h\alpha^2}-1}{h} \right\} .$$

Since

$$\left|\phi(\alpha)e^{-y\alpha^2} \frac{e^{-h\alpha^2}-1}{h}\right| \leq |\phi(\alpha)| e^{-y\alpha^2}\alpha^2 ,$$

as $h\to 0$, the transforms of (7.4a) are convergent in L_2 norm towards the transform of the function $\phi(\alpha)e^{-y\alpha^2}(-\alpha^2)$, namely

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x}\phi(\alpha)e^{-y\alpha^2}(-\alpha^2)d\alpha$$

which is the literal derivative $\frac{\partial f(x,y)}{\partial y}$, for $y > 0$, and this establishes (7.4).

(7.5) for every $y > 0$, $\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial y}$

[by direct computation].

(7.6) $$\left(\int_{-\infty}^{\infty} |f(x,y)|^2 dx\right)^{\frac{1}{2}} \leq M = \| f \| .$$

<u>THEOREM 68</u>. <u>For given</u> $f(x) \in L_2$ <u>any</u> $f(x,y)$ <u>with properties</u> (7.1) - (7.6) <u>must be the function given by the formula</u> (7*).

Proof: Since $f(x,y) \in L_2$ as a function in x, by (7.1), it has a transform $\phi(\alpha,y) \in L_2$. Because of (7.2),

$$\phi(\alpha,y) \to \phi(\alpha) \text{ in } L_2 \text{ norm}$$

as $y \to 0$. Since, by (7.3), $\frac{\partial^2 f}{\partial x^2} \in L_2$, we have

$$\bar{T}[\frac{\partial^2 f}{\partial x^2}] = (-i\alpha)^2 \phi(\alpha,y)$$

by theorem 61. Furthermore

$$\bar{T}[\frac{f(x,y+h)-f(x,y)}{h}] = \frac{\phi(\alpha,y+h)-\phi(\alpha,y)}{h},$$

and therefore by Plancherel's theorem and (7.4), there exists $\frac{\partial\phi(\alpha,y)}{\partial y} \in L_2$ such that

$$\frac{\partial\phi(\alpha,y)}{\partial y} = \bar{T}[\frac{\partial f}{\partial y}].$$

However

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial y}$$

by (7.5), and hence we obtain

$$\frac{\partial\phi(\alpha,y)}{\partial y} = \bar{T}[\frac{\partial f}{\partial y}] = \bar{T}[\frac{\partial^2 f}{\partial x^2}] = (-i\alpha)^2 \phi(\alpha,y).$$

Therefore

$$\phi(\alpha,y) = A(\alpha)\, e^{-\alpha^2 y},$$

except on an α-null set common to all y. We know that

$$\phi(\alpha,y) \to \phi(\alpha) \text{ in } L_2\text{-norm}$$

as $y \to 0$. But $A(\alpha)e^{-\alpha^2 y}$ converges to $A(\alpha)$ in L_2-norm as $y \to 0$. Hence,

$$\phi(\alpha) = A(\alpha)$$

for almost all α . That is

$$\phi(\alpha, y) = \phi(\alpha)e^{-\alpha^2 y} .$$

However, the function given by (7*) has the same transform, namely $\phi(\alpha)e^{-\alpha^2 y}$. Hence $f(x,y)$ is equal to (7*).

Instead of $K(x) = e^{-x^2}$, we can consider $K(x) = e^{-|x|}$. Accordingly, let $f(x) \in L_2$, $\bar{T}f = \phi(\alpha)$ and

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\alpha) e^{-ix\alpha - y|\alpha|} d\alpha$$

(7**) $$= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(z) \frac{y}{y^2+(y-z)^2} dz .$$

It is not hard to verify the following

<u>Properties of $f(x,y)$:</u>

(7.1)' $f(x,y) \in L_2$ as a function in x.

(7.2)' $f(x,y) \to f(x)$ in L_2-norm as $y \to +0$.

(7.3)' $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and belong to L_2 .

(7.4)' $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ exist in the same manner as in (7.4) and $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial f}{\partial y})$.

(7.5)' For every $y > 0$, $-\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2}$

(7.6)' $$\left(\int_{-\infty}^{\infty} |f(x,y)|^2 dx\right)^{\frac{1}{2}} \leq M = \|f\| .$$

THEOREM 69. *For given* $f(x) \in L_2$ *any function* $f(x,y)$ *with properties* (7.1)'-(7.6)' *must be the function given by* (7**).

Proof: Let $\overline{T}[f(x,y)] = \phi(\alpha,y) \in L_2$. By (7.3)' and theorem 61,

$$\overline{T}[\frac{\partial^2 f}{\partial x^2}] = -\alpha^2 \phi(\alpha,y) .$$

Since $\overline{T}[\frac{f(x,y+h)-f(x,y)}{h}] = \frac{\phi(\alpha,y+h)-\phi(\alpha,y)}{h}$, and because of (7.4)', there exists $\frac{\partial\phi(\alpha,y)}{\partial y} \in L_2$ such that

$$\frac{\partial\phi(\alpha,y)}{\partial y} = \overline{T}[\frac{\partial f}{\partial y}],$$

and

$$\frac{\partial^2\phi(\alpha,y)}{\partial y^2} = \overline{T}[\frac{\partial^2 f}{\partial y^2}].$$

Because of (7.5)', it follows that

$$\frac{\partial^2\phi(\alpha,y)}{\partial y^2} = \alpha^2 \phi(\alpha,y)$$

so that

(7.7) $$\phi(\alpha,y) = A(\alpha)e^{-\alpha y} + B(\alpha)e^{\alpha y}$$

which converges in L_2-norm towards $A(\alpha) + B(\alpha)$, as $y \to 0$. Hence

(7.8) $$\phi(\alpha) = A(\alpha) + B(\alpha)$$

almost everywhere. Solving (7.7) and (7.8) for $A(\alpha)$ and setting $y = 1$, we obtain,

$$A(\alpha) = \frac{\phi(\alpha,1) - \phi(\alpha)e^{\alpha}}{e^{-\alpha} - e^{\alpha}}$$

Hence in the interval $0 < a < \alpha < b$, $A(\alpha) \in L_2$. Therefore from (7.7) we obtain

$$\int_a^b |B(\alpha)|^2 \, d\alpha \leq \int_a^b |\phi(\alpha,y)e^{-\alpha y}|^2 \, d\alpha$$

$$+ \int_a^b |A(\alpha)e^{-2\alpha y}|^2 \, d\alpha$$

$$< M e^{-ay} + N e^{-2ay} .$$

Letting $y \to \infty$, we see that $B(\alpha) = 0$ almost everywhere for $\alpha > 0$; similarly $A(\alpha) = 0$ almost everywhere for $\alpha < 0$. Because of (7.8), it follows that

$$\phi(\alpha,y) = \phi(\alpha)e^{-|\alpha|y}$$

which is the same as the transform of the function given by formula (7**). Thus any $f(x,y)$ with properties (7.1)' - (7.6)' must be equal to (7**).

§8. Simple type of bounded transformation

If $W(\alpha)$ is a bounded measurable function in $-\infty < \alpha < \infty$, and

(8.1) $$|W(\alpha)| \leq M, \quad \alpha \in (-\infty,\infty)$$

then for every fixed $p \geq 1$, for $\phi(\alpha) \in L_p(-\infty,\infty)$, the function

$$\psi(\alpha) = W(\alpha) \, \phi(\alpha)$$

belongs again to $L_p(-\infty,\infty)$, since

$$\left(\int |\psi(\alpha)|^p \, d\alpha\right)^{\frac{1}{p}} \leq M \left(\int_{-\infty}^{\infty} |\phi(\alpha)|^p \, d\alpha\right)^{\frac{1}{p}} .$$

Thus,

(8.2) $$T \, \phi(\alpha) = W(\alpha) \, \phi(\alpha)$$

is a bounded linear transformation from $L_p(-\infty,\infty)$ into itself, and

(8.3) $$|T|_p \leq M .$$

Obviously, we may put for M the smallest number compatible with (8.1), and since the elements of L_p are determined only up to a set of measure zero, we have

(8.4) $$|T|_p \leq \text{ess. u.b. } |W(\alpha)| = M_o .$$

Actually in (8.4) we have the sign of equality. In fact, for any $\epsilon > 0$, there exists a set A_ϵ of positive, finite measure, for which

$$|W(\alpha)| \geq M_o - \epsilon, \quad \alpha \in A_\epsilon .$$

Now, define

$$\phi_\epsilon(\alpha) = \begin{cases} 1, & \alpha \in A_\epsilon , \\ 0, & \alpha \notin A_\epsilon . \end{cases}$$

Then, if $T_\epsilon(\alpha) = T\phi_\epsilon(\alpha)$, we have

$$\left(\int_{-\infty}^{\infty} |T_\epsilon(\alpha)|^p \, d\alpha\right)^{\frac{1}{p}} = \left(\int_{A_\epsilon} |W(\alpha)|^p \, d\alpha\right)^{\frac{1}{p}}$$

$$\geq (M_o - \epsilon) \int_{A_\epsilon} d\alpha = (M_o - \epsilon) \| \phi_\epsilon \|_p .$$

We will now restrict ourselves to the case p = 2. In this case, any operation of the form (8.2) will be called *a transformation of simple type*. Such transformations satisfy the following

THEOREM 70. *If* $\psi(\alpha) = T\,\phi(\alpha)$ *is of simple type, then whenever the element* $\phi(\alpha)$ *vanishes over a measurable set A, its transform* $\psi(\alpha)$ *also vanishes there. Conversely, if for a bounded linear transformation* $\psi(\alpha) = T\,\phi(\alpha)$, *the vanishing of an element* $\phi(\alpha)$ *outside a bounded interval (a,b) implies the vanishing of* $T\,\phi(\alpha)$ *outside that interval, then the transformation is of simple type.*

Proof: The first part of the theorem is quite obvious. Now, for the proof of the converse introduce the functions

$$\phi_{a,b}(\alpha) = \begin{cases} 1, & a \leq \alpha \leq b \\ 0, & \text{outside } [a,b] \end{cases}$$

and put

$$\psi_{ab}(\alpha) = T\,\phi_{ab}(\alpha) .$$

Then for $a_2 \leq a_1 < b_1 \leq b_2$, we have

$$T[\phi_{a_2b_2}(\alpha) - \phi_{a_1b_1}(\alpha)] = \psi_{a_2b_2}(\alpha) - \psi_{a_1b_1}(\alpha) . \tag{8.5}$$

Since T is linear and since

$$\phi_{a_2b_2}(\alpha) - \phi_{a_1b_1}(\alpha)$$

vanishes in $a_1 \leq \alpha \leq b_1$, the right side of (8.5) also vanishes there by hypothesis. Hence there exists a function $W(\alpha)$ in $-\infty < \alpha < \infty$ such that for any $a_1 < \alpha < b_1$, we have

$$\psi_{a_1 b_1}(\alpha) = \begin{cases} W(\alpha), & a_1 \leq \alpha < b_1 \\ 0 \quad, & \text{outside } [a_1, b_1). \end{cases}$$

Furthermore,

$$\int_{-\infty}^{\infty} |\psi_{a_2 b_2}(\alpha) - \psi_{a_1 b_1}(\alpha)|^2 \, d\alpha = \int_{a_2}^{a_1} |W(\alpha)|^2 \, d\alpha + \int_{b_1}^{b_2} |W(\alpha)|^2 d\alpha$$

and

$$\int_{-\infty}^{\infty} |\phi_{a_2 b_2}(\alpha) - \phi_{a_1 b_1}(\alpha)|^2 d\alpha = (a_1 - a_2) + (b_2 - b_1).$$

Thus by the boundedness of T we have

$$\int_{a_2}^{a_1} |W(\alpha)|^2 d\alpha + \int_{b_1}^{b_2} |W(\alpha)|^2 d\alpha \leq M \{(a_1 - a_2) + (b_2 - b_1)\}.$$

Putting $a_1 = a_2$, we obtain

$$\frac{1}{b_2 - b_1} \int_{b_1}^{b_2} |W(\alpha)|^2 d\alpha \leq M \; .$$

Hence

$$|W(\alpha)| \leq M$$

almost everywhere. Thus, for a special function $\phi_{ab}(\alpha)$, we obtain

$$T \phi_{ab}(\alpha) = W(\alpha) \; \phi_{ab}(\alpha), \quad |W(\alpha)| \leq M.$$

Now, a step-function $\phi(\alpha)$ is a finite linear combination of functions ϕ_{ab}. Hence we have for step functions

(8.6) $$T \; \phi(\alpha) = W(\alpha)\phi(\alpha)$$

with

(8.7) $$|W(\alpha)| \leq M \; .$$

Now, both sides of (8.6) can be formed for arbitrary elements of $L_2(-\infty, \infty)$, and it follows easily, from L_2-approximation by step-functions, that (8.6) holds always as claimed.

Addendum to theorem 70: In the converse part of theorem 70 it suffices to make the hypothesis only for intervals of the form $(-A < \alpha < A)$. Pursuing the reasoning, we obtain

$$\int_{-A_2}^{-A_1} |W(\alpha)|^2 \, d\alpha + \int_{A_1}^{A_2} |W(\alpha)|^2 \, d\alpha \leq 2M\,(A_2 - A_1).$$

From this follows

$$\frac{1}{A_2 - A_1} \int_{A_1}^{A_2} |W(\alpha)|^2 \, d\alpha \leq 2M ;$$

hence the conclusion $|W(\alpha)| \leq 2M$, with the factor 2, at first; but once the conclusion has been reached, the bound can be reduced to M itself.

§9. Bounded transformations commutative with translations

Before we proceed further, we observe that if T^x is a bounded linear transformation on $L_2(-\infty, \infty)$ which carries f into g:

$$T^x f(x) = g(x)$$

there corresponds a bounded linear transformation T^α which carries the Fourier transform of f into the Fourier transform of G:

$$T^\alpha \phi(\alpha) = \psi(\alpha)$$

and conversely.

This follows from the fact that the Fourier transformation in L_2 is isometric. We have only to define

$$g(x) = T^x f(x) = \underset{A \to \infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-i\alpha x}\, T^{\alpha}\phi(\alpha)d\alpha$$

so that

$$\|g\| \leq M\|\phi\| = M\|f\| .$$

THEOREM 71. *If* $f(x) \in L_2$, $g(x) \in L_2$ *and* $\phi(\alpha), \psi(\alpha)$ *are the Fourier transforms of* f *and* g *respectively, and if*

$$T^{\alpha}\phi(\alpha) = W(\alpha)\phi(\alpha) = \psi(\alpha), \qquad |W(\alpha)| < M ,$$

and if we introduce for any h *the operation,*

$$R_h[f(x)] = f(x+h)$$

(which we may call translation by h*), then*

$$T^x R_h[f(x)] = R_h T^x[f(x)] .$$

In other words, such a transformation T^{α} *is commutative with translations.*

Proof:

$$R_h[f(x)] = \underset{A \to \infty}{\text{l.i.m}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-ix\alpha}\,(e^{-i\alpha h}\phi(\alpha))d\alpha .$$

Therefore,

$$T^x R_h[f(x)] = \underset{A \to \infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-ix\alpha}\, T^{\alpha}[e^{-i\alpha h}\,\phi(\alpha)]d\alpha$$

$$= \underset{A \to \infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-i\alpha(x+h)} W(\alpha)\phi(\alpha)d\alpha$$

$$= g(x+h).$$

We shall next establish the converse of theorem 71.

<u>THEOREM 72</u>. <u>If</u> T^x <u>is a bounded linear transformation such that</u> $T^x[f(x+h)] = g(x+h)$ <u>then there exists a function</u> $W(\alpha)$, $|W(\alpha)| < M$ <u>such that</u>

$$\psi(\alpha) = T^\alpha \phi = W(\alpha)\phi(\alpha).$$

Proof: The hypothesis implies that

$$T^\alpha[\phi(\alpha)e^{-i\alpha h}] = e^{-i\alpha h}\,\psi(\alpha) = e^{-i\alpha h}\,T^\alpha \phi(\alpha) .$$

Since T^α is linear, it follows that

$$T^\alpha[\phi(\alpha)\ P_n(\alpha)] = P_n(\alpha)\ T^\alpha\ \phi(\alpha)$$

where $P_n(\alpha)$ is an exponential polynomial.

Next, if $P(\alpha)$ is a bounded periodic function, then there exists a sequence of exponential polynomials $P_n(\alpha)$ such that

(9.1) $$|P_n(\alpha) - P(\alpha)| < c$$

independently of α and n, and

(9.2) $$\lim_{n\to\infty} |P_n(\alpha) - P(\alpha)| = 0$$

for almost all α. For instance, the Fejer polynomials are of this kind.

Since $\phi(\alpha) \in L_2$, it follows that

$$P_n(\alpha)\phi(\alpha) \to P(\alpha)\phi(\alpha)$$

in L_2-norm. Hence

$$T[P_n(\alpha)\phi(\alpha)] \to T[P(\alpha)\phi(\alpha)] .$$

However, by the same argument,

$$P_n(\alpha).T\phi(\alpha) \to P(\alpha).T\phi(\alpha) .$$

Hence

$$T[P(\alpha)\phi(\alpha)] = P(\alpha)T\phi(\alpha).$$

Next let us consider a function $Q(\alpha)$ such that

$$|Q(\alpha)| \leq M , \qquad \alpha \in (a,b);$$
$$Q(\alpha) = 0 , \qquad \alpha \notin (a,b) .$$

Construct the functions

$$Q_n(\alpha) = \begin{cases} Q(\alpha), & \alpha \in (a,b) \\ 0 , & \alpha \in (-l_n,a), \\ & \text{or } \alpha \in (b,l_n), \end{cases}$$

and $Q_n(\alpha)$ repeats itself periodically with period 2π . Then the sequence $\{Q_n(\alpha)\}$ satisfies conditions analogous to (9.1) and (9.2), and in an analogous way it follows that

$$Q_n(\alpha)\phi(\alpha) \to Q(\alpha)\phi(\alpha)$$

in L_2-norm, and hence

$$T[Q(\alpha)\phi(\alpha)] = Q(\alpha).T\phi(\alpha) .$$

Suppose $\phi(\alpha) \neq 0$ on a <u>bounded</u> set S , and $Q(\alpha)$ is the characteristic function of S. Let $S'(= E_1 - S)$ denote the complement of S. Then, for $\alpha \in S'$, we see that $T[\phi(\alpha)] = 0$

since $\phi(\alpha) = 0$ in S'.

Next, take an arbitrary set S, and a sequence of bounded sets $\{S_n\}$ such that

$$0 \subset S_1 \subset S_2 \ldots \subset S_n \to S .$$

Let $\phi(\alpha) \neq 0$ on S. Construct the functions

$$\phi_n(\alpha) = \begin{cases} \phi(\alpha) , & \alpha \in S_n \\ 0 , & \alpha \notin S_n' . \end{cases}$$

Since the function $\phi_n(\alpha)$ vanishes on S_n' which is the complement of a bounded set, it follows that $T[\phi_n(\alpha)] = 0$, $\alpha \in S_n'$, and a fortiori for $\alpha \in S'$; since $\phi_n(\alpha) \to \phi(\alpha)$ in L_2-norm, it follows that $T[\phi_n(\alpha)] \to T[\phi(\alpha)]$ and hence $T[\phi(\alpha)] = 0$ for $\alpha \in S'$. Thus $\phi(\alpha) = 0$ in S' implies $T[\phi(\alpha)] = 0$ in S'. By an application of theorem 70 the result follows:

THEOREM 73. *If T^x is a bounded linear operator on $L_2(-\infty,\infty)$ so that*

$$T^x f(x) = g(x) ,$$

and $T^x R_h[f(x) = R_h . T^x[f(x)]$; *then whenever* $f(x)$ *has a derivative* $Df(x) \in L_2$, $T^x[f]$ *also has a derivative, and*

$$T^x\, Df(x) = D . T^x f(x).$$

Proof: By hypothesis

$$T^x\left[\frac{f(x+h)-f(x)}{h}\right] = \frac{g(x+h)-g(x)}{h} .$$

By theorem 66,

$$\frac{f(x+h)-f(x)}{h} \to Df(x)$$

in L_2-norm; because of continuity,

$$\frac{g(x+h)-g(x)}{h} \to T^x[Df(x)]$$

in L_2-norm. By theorem 66, it follows that

$$T^x[Df(x)]$$

is the derivative of g(x) almost everywhere.

THEOREM 74. If T^x is a bounded linear operator on $L_2(-\infty,\infty)$ such that $T^x f(x) = g(x)$, and if

$$T^x[Df(x)] = D[T^x f(x)],$$

whenever the two derivatives exist, then

$$T^x R_h[f(x)] = R_h . T^x[f(x)] .$$

Proof:

Let ϕ, ψ be the Fourier transforms of f and g respectively; then $T^\alpha \phi = \psi$. Consider a function defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(\alpha) e^{-i\alpha x}\, d\alpha$$

for a finite A ; differentiating n times, we have

$$f^{(n)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} (-i\alpha)^n \phi(\alpha) e^{-i\alpha x}\, d\alpha$$

If $T^x[f^{(n)}(x)] = g^{(n)}(x)$, then

$$g^{(n)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} (-i\alpha)^n \psi(\alpha) e^{-i\alpha x}\, d\alpha$$

where ψ is the Fourier transform of g. Since T^x is bounded

$$\int_{-\infty}^{\infty} |\alpha|^{2n} \cdot |\psi(\alpha)|^2 \, d\alpha \leq M \int_{-\infty}^{\infty} |\phi(\alpha)|^2 |\alpha|^{2n} \, d\alpha .$$

Let $\phi(\alpha) = 0$ outside $(-A,A)$; then

$$\int_{-\infty}^{\infty} |\alpha|^{2n} \cdot |\psi(\alpha)|^{2n} \, d\alpha \leq M \int_{-A}^{A} |\phi(\alpha)|^2 |\alpha|^{2n} \, d\alpha$$

$$\leq C \, A^{2n} ;$$

from this it follows that $\psi(\alpha)$ is essentially zero outside $(-A,A)$. Hence, applying addendum to theorem 70, we infer that T^{α}, and so T^{x}, is of simple type. Using theorem 71 now we obtain the required result.

§10. Closure of translations

Take a function $f(x) \in L_2$, with Fourier transform $\phi(\alpha)$ and denote by S_ϕ a measurable set in E_1 such that, except for null sets, $\phi(\alpha) \neq 0$ in S_ϕ and $\phi(\alpha) = 0$ in $E_1 - S_\phi$. If

(10.1) $$f_1(x) = c_1 f(x+t_1) + \ldots + c_n f(x+t_n)$$

is an arbitrary, finite, linear combination which does not vanish identically, then its Fourier transform is

$$\phi_1(\alpha) = (c_1 e^{-i\alpha t_1} + \ldots + c_n e^{-i\alpha t_n}) \, \phi(\alpha)$$

and

$$S_{\phi_1} = S_\phi$$

since the exponential factor has discrete zeros at most.

Now form the L_2 closure of all linear combinations (10.1) denoting the family by H(f). Obviously, for $f^* \in H(f)$ we have $S_{\phi^*} \subset S_\phi$. But the converse is also true. Thus we have the following

THEOREM 75. *If* $g(x) \in L_2(E_1)$ *and* $\overline{T}g = \psi(\alpha)$, *and if*

$$(10.2) \qquad S_\psi \subset S_\phi$$

then $g \in H(f)$.

Proof: For any $g \in L_2$, there exists an element f^0 in H(f) such that for $g^0 = g - f^0$ we have

$$(10.3) \qquad \int_{-\infty}^{\infty} g^0(x)\overline{f^*(x)}\, dx = 0$$

for all $f^* \in H(f)$. Now, for the transform ψ^0 of g^0 we again have (10.2). We apply (10.3) for the special function $f^*(x) = f(x+t)$, t being any fixed number. By convolution (10.3) amounts to

$$\int_{-\infty}^{\infty} \psi^0(\alpha)\overline{\phi(\alpha)}e^{i\alpha t}\, d\alpha = 0 \ .$$

Now $\psi^0(\alpha)\overline{\phi(\alpha)} \in L_1(-\infty,\infty)$; hence by theorem 5,

$$(10.4) \qquad \psi^0(\alpha)\overline{\phi(\alpha)} = 0$$

almost everywhere. Since it is known that $\psi^0(\alpha) = 0$ in $E_1 - S_\phi$, (10.4) implies $\psi^0(\alpha) = 0$ almost everywhere and thus $g^0(x) = 0$ almost everywhere. Hence $g(x) = f^0(x)$ where $f^0(x) \in H(f)$, as claimed. In particular, we have

THEOREM 76. *If* $f(x) \in L_2$ *and* $\overline{T}[f] = \phi(\alpha) \neq 0$ *almost everywhere, then every* $g \in L_2$ *belongs to the* L_2*-closure of* (10.1).

CHAPTER V.

GENERAL TRANSFORMS IN L_2

§1. General unitary transformations in $L_2(0,\infty)$

In the previous chapter we have seen that the Fourier transform represents a unitary transformation in $L_2(E_k)$; this representation was obtained in Plancherel's theorem. It is our object now to consider the most general unitary transformations in $L_2(0,\infty)$. If $f(x) \in L_2(0,\infty)$ and T is a unitary transformation in $L_2(0,\infty)$ so that $Tf = g$, we shall show that T can be represented in the form

$$\int_0^a g(x)dx = \int_0^\infty \overline{k(a,y)}f(y)dy ,$$

and its inverse, T^{-1}, can be represented as

$$\int_0^a f(x)dx = \int_0^\infty \overline{l(a,y)}g(y)dy ,$$

where k and l are well defined, and will be characterized uniquely.

Suppose T is a given unitary transformation in L_2 $(0,\infty)$ and $Tf(x) = g(y)$; then, plainly, $T^{-1}[g(y)] = f(x)$.

Define, for every fixed $a > 0$,

$$g_a(y) = \begin{cases} 1 , & 0 < y < a , \\ 0 , & y \geq a . \end{cases}$$

Then $g_a(y) \in L_2(0,\infty)$. Define,

$$k(a,x) = T^{-1}[g_a(y)],$$

the inverse existing because T is unitary. Then,

$$T[k(a,x)] = g_a(y)$$

and

$$\int_0^\infty g(y)g_a(y)dy = \int_0^\infty \overline{k(a,x)}f(x)dx, \tag{1.1}$$

since T preserves inner products; therefore

$$\int_0^a g(y)dy = \int_0^\infty \overline{k(a,x)}\, f(x)dx. \tag{1.2}$$

Now, define

$$l(a,y) = T[g_a(x)] ;$$

then in a similar fashion, we obtain

$$\int_0^b f(x)dx = \int_0^\infty \overline{l(b,y)}g(y)dy . \tag{1.3}$$

Also, from (1.1) we obtain,

$$\int_0^\infty g_a(x)g_b(x)dx = \int_0^\infty k(a,x)\,\overline{k(b,x)}dx ,$$

and hence

$$\int_0^\infty k(a,x)\,\overline{k(b,x)}dx = \min(a,b) , \tag{1.4}$$

and similarly

$$\int_0^\infty l(a,y)\,\overline{l(b,y)}\,dy = \min(a,b). \tag{1.5}$$

In general,

$$\int_0^\infty f_1\overline{f}_2\,dx = \int_0^\infty Tf_1 \,.\, \overline{Tf_2}\,dy ,$$

and choosing $f_1 = g_b(x)$, $f_2 = k(a,x)$, we have, $Tf_1 = l(b,x)$ and $Tf_2 = g_a(x)$ and therefore

$$(1.6) \qquad \int_0^b \overline{k(a,x)}dx = \int_0^a l(b,x)dx .$$

Thus we are led to the following

<u>THEOREM 77</u>. <u>To each unitary transformation</u> T <u>in</u> $L_2(0,\infty)$, <u>there correspond two functions</u> $k(a,x)$, $l(a,x)$ <u>belonging to the class</u> $L_2(0,\infty)$ <u>for each</u> a, $0 < a < \infty$, <u>such that</u> (1.2) - (1.6) <u>hold; the functions</u> $k(a,x), l(a,x)$ <u>so determined are unique</u> (<u>neglecting null sets</u>).

Proof: It remains for us to prove only the uniqueness of k and l, because the other properties have already been obtained. For given T, take any other function $\widetilde{k}$, which satisfies (1.2) - (1.6). Then, from (1.2), we will have

$$\int_0^a g(y)dy = \int_0^\infty \overline{k(a,x)}f(x)dx = \int_0^\infty \overline{\widetilde{k}(a,x)}f(x)dx$$

so that

$$\int_0^\infty \overline{[k(a,x) - \widetilde{k}(a,x)]}f(x)dx = 0$$

for every f; in particular, if $f = k(a,x) - \widetilde{k}(a,x)$, we obtain:

$$\| k(a,x) - \widetilde{k}(a,x)\| = 0$$

for each a; hence $k(a,x) = \widetilde{k}(a,x)$ for almost all x, and similarly for l.

We shall now prove a result which is, in a sense, the converse of theorem 77.

THEOREM 78. *Given the functions (or kernels) $k(a,x)$, $l(a,x)$ belonging to $L_2(0,\infty)$ for each $a > 0$, and satisfying* (1.4), (1.5) *and* (1.6), *then* (1.2) *and* (1.3) *define a unitary transformation $f \rightarrow g$ in $L_2(0,\infty)$ and its inverse respectively.*

Proof: We shall define the operators U and V for the function $g_a(x)$ as follows:

$$Ug_a(x) = k(a,y)$$

$$Vg_a(x) = l(a,y).$$

On account of hypothesis (1.4), U and V preserve inner products, so that

(1.7) $$(Ug_a, Ug_b) = (g_a, g_b)$$

and

(1.8) $$(Vg_a, Vg_b) = (g_a, g_b) .$$

Furthermore,

(1.9) $$(g_b, Ug_a) = (Vg_b, g_a) .$$

(1.7) shows that on the space of functions of the form $g_a(y)$, the operator U is isometric, and so is V by (1.8). If we now *define*

$$g_{ab}(y) = g_b(y) - g_a(y) ,$$

then, any step-function g may be written as

$$g = \sum c_n \, g_{a_n b_n}(y) = \sum d_n g_n \text{ , say.}$$

By the linearity of the definite integrals, it is easily seen that in relations (1.7) to (1.9) we may replace first g_a and then g_b by arbitrary step-functions, and it follows therefore that the operators U and V are isometric over the class of step-functions which are dense in L_2. We can therefore extend the operator to the whole of L_2; if the extended operator is $\bar{U}$, then $\bar{U}$ is also isometric. Hence

$$(1.10) \qquad (\bar{U}f, \bar{U}g) = (f, g)$$
$$(1.11) \qquad (\bar{V}f, \bar{V}g) = (f, g) \qquad f, g \in L_2(0,\infty) .$$

Again, if g_n is a step-function, and $\{g_n\} \to g$ in L_2-norm, and f is a fixed step-function, then

$$(Vf, g_n) = (f, Ug_n), \quad \text{by (1.9)},$$

and letting $n \to \infty$, we obtain

$$(Vf, g) = (f, \bar{U}g).$$

We can similarly replace the step-function f by an arbitrary function $f \in L_2(0,\infty)$ and finally obtain

$$(1.12) \qquad (\bar{V}f, g) = (f, \bar{U}g) , \quad f, g \in L_2(0,\infty).$$

To show that $\bar{U}$ and $\bar{V}$ are inverses of each other, let ϕ and ψ be arbitrary functions in $L_2(0,\infty)$; then by (1.12),

$$(\phi, \bar{U}(\bar{V}\phi)) = (\bar{V}\phi, \bar{V}\psi)$$

$$= (\phi, \psi) \quad \text{by } (1.11).$$

For a fixed ψ, it follows that

$$(\phi, \psi - \bar{U}(\bar{V}\psi)) = 0$$

for <u>every</u> ϕ. Hence $\psi = \bar{U}(\bar{V}\psi)$ almost everywhere. That is, for any ψ, there exists an element $\chi = \bar{V}\psi$ such that $\psi = \bar{U}\chi$. In other words, $\bar{U}$ has an inverse. Similarly $\bar{V}$ has an inverse, and since $\bar{U}\bar{V} = 1$ it follows that $\bar{U}$ and $\bar{V}$ are inverses of each other; taken in conjunction with (1.10) - (1.12), it follows that $\bar{U}$ is a unitary transformation. That this unitary transformation can be represented as in (1.2) and (1.3) follows as before (see Theorem 77).

<u>Remarks</u>: Suppose that $k(a,x)$ has a partial derivative $k_1(a,x) = \frac{\partial}{\partial a}[k(a,x)]$ in the following sense: there exists $k_1(a,x)$ continuous in $0 \leq a < \infty$ and $0 \leq x < \infty$ such that

$$k(a,x) = \int_0^a k_1(\alpha,x)d\alpha .$$

Then (1.2) can also be read as

$$g(y) = \underset{n\to\infty}{\text{l.i.m.}} \int_0^n k_1(y,x)f(x)dx. \tag{1.13}$$

For, set

$$f_n(x) = \begin{cases} f(x), & 0 \leq x < n, \\ 0, & x \geq n. \end{cases}$$

Then $f_n \to f$ in L_2-norm; and if we define

$$g_n(y) = \int_0^\infty k_1(y,x) f_n(x)dx$$

then

$$\int_0^a g_n(y)dy = \int_0^\infty k(a,x) f_n(x)dx \ .$$

Hence

$$g_n(y) = Tf_n(x)$$

where T is continuous; therfore $Tf_n \to Tf$ in L_2-norm; in other words, $g_n \to g = Tf$, from which (1.13) follows.

§2. Watson transforms

Assume that there exist functions $k(x)$, $l(x)$ defined everywhere in $0 < x < \infty$, such that

(2.1) $$k(a,x) = \frac{k(ax)}{x}, \quad l(a,x) = \frac{l(ax)}{x} \ ;$$

then (1.6) reads

(2.2) $$\int_0^{ab} \frac{\overline{k(x)}}{x} dx = \int_0^{ab} \frac{l(x)}{x} dx \ ,$$

and therefore $l(x) = \overline{k(x)}$ essentially, and (1.4) and (1.5) reduce to the single relation

(2.3) $$\int_0^\infty \frac{k(ax)\overline{k(bx)}}{x^2} dx = \min(a,b) \ .$$

Conversely, starting with a function $k(x)$ in $0 < x < \infty$, we define $k(a,x)$ and $l(a,x)$ by (2.1), then (2.3) implies (1.4), (1.5) and (1.6). Hence we have

<u>THEOREM 79</u>. <u>Given</u> $\frac{k(x)}{x} \in L_2(0,\infty)$ <u>and such that</u>

$$\int_0^\infty \frac{k(ay)\overline{k(by)}}{y^2}\, dy = \min(a,b) ,$$

<u>there exists a transformation</u> T <u>in</u> $L_2(0,\infty)$ <u>which is unitary, and if</u> Tf = g, <u>then</u>

(2.4) $$\int_0^a g(x)dx = \int_0^\infty \frac{\overline{k(ay)}}{y} f(y)dy$$

(2.5) $$\int_0^a f(x)dx = \int_0^\infty \frac{k(ay)}{y} g(y)dy .$$

This transformation is called a <u>Watson transform</u>.

We will now rewrite the Watson transform from $(0,\infty)$ into $(-\infty,\infty)$. If we set:

$$x = e^Y,\ y = e^Y;\ F(Y) = f(e^Y)e^{Y/2},\ G(Y) = g(e^Y)e^{Y/2} ,$$

$$K(Y) = \frac{k(e^Y)}{e^{Y/2}},\quad A = \log a,\quad B = \log b,$$

then we can rewrite the conditions of theorem 79 as follows:

(2.6) $$e^{-A/2}\int_{-\infty}^A G(Y)e^{Y/2}dY = \int_{-\infty}^\infty \overline{K(Y+A)}\, F(y)dY$$

(2.7) $$e^{-A/2}\int_{-\infty}^A F(Y)e^{Y/2}dY = \int_{-\infty}^\infty K(Y+A)G(Y)dY$$

$$e^{\frac{A+B}{2}}\int_{-\infty}^\infty K(Y+A)\overline{K(Y+B)}dY = \min(e^A,\ e^B)$$

or, $$\int_{-\infty}^\infty K(Y+A)\overline{K(Y+B)}dY = \min(e^{\frac{A-B}{2}},\ e^{-\frac{A}{2}+\frac{B}{2}})$$

(2.8) $$= e^{-\left|\frac{A-B}{2}\right|}$$

(2.9) $$\int_{-\infty}^{\infty} |F(Y)|^2 \, dY = \int_{-\infty}^{\infty} |G(Y)|^2 \, dY$$

(2.10) $$\int_{-\infty}^{\infty} |K(Y)|^2 \, dY < \infty .$$

We shall now reformulate theorem 79 as follows:

THEOREM 79'. Any Watson transform is isomorphic to the following transformation: take any function $K(Y) \in L_2(-\infty,\infty)$ with the property (2.8) which automatically implies (2.10); then corresponding to each $F(Y) \in L_2(-\infty,\infty)$ there exists a unique $G(Y) \in L_2(-\infty,\infty)$ such that (2.3), (2.4) and (2.6) hold.

Remarks: On the other hand, it is much easier to see that if we are given a function $K(Y) \in L_2$ such that (2.6), (2.7) and (2.9) hold, then (2.8) must necessarily be satisfied. This follows from theorem 77 with little difficulty. Hence we can phrase theorem 79' in a slightly more comprehensive form:

THEOREM 79". If $K(Y) \in L_2(-\infty,\infty)$, then in order that (2.6), (2.7) and (2.9) may hold, it is necessary and sufficient that

$$\int_{-\infty}^{\infty} K(Y+A)\overline{K(Y+B)}dY = e^{-\left|\frac{A-B}{2}\right|}$$

for all values of A and B in $(0,\infty)$.

Since F, G, K belong to $L_2(-\infty,\infty)$, by Plancherel's theorem we have their Fourier transforms $\phi, \psi, R \in L_2(-\infty,\infty)$ such that

$$(2.11) \qquad F(Y) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iY\alpha}\, \phi(\alpha) d\alpha$$

$$(2.12) \qquad G(Y) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iY\alpha}\, \psi(\alpha)\, d\alpha$$

$$(2.13) \qquad K(Y) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iY\alpha}\, R(\alpha) d\alpha\ .$$

Also by (2.9), (2.11) and (2.12), we have

$$(2.14) \qquad \int_{-\infty}^{\infty} |\phi(\alpha)|^2\, d\alpha = \int_{-\infty}^{\infty} |\psi(\alpha)|^2 d\alpha$$

and from (2.8) we have

$$\int_{-\infty}^{\infty} |R(\alpha)|^2\, e^{-i\alpha(A-B)}\, d\alpha = e^{-|\frac{A-B}{2}|}$$

or,

$$\int_{-\infty}^{\infty} |R(\alpha)|^2\, e^{-i\alpha y}\, d\alpha = e^{-|\frac{y}{2}|}.$$

Hence

$$(2.15) \qquad |R(\alpha)|^2 = \frac{1}{2\pi} \cdot \frac{1}{\frac{1}{4} + \alpha^2}\ .$$

§3. Functional equation associated with Watson transforms

Let us now consider the left side of equation (2.7) with x in place of A:

$$(3.1) \qquad H(x) = e^{-x/2} \int_{-\infty}^{x} F(y) e^{y/2}\, dy, \quad -\infty < x < \infty.$$

The integral in (3.1) exists in the ordinary sense because

$$(3.1)' \quad \int_{-\infty}^{x} |F(y)e^{y/2}|^2\,dy \leq \int_{-\infty}^{x} |F(y)|^2\,dy \cdot \int_{-\infty}^{x} e^{y}\,dy < \infty .$$

We wish to analyze H(x) directly so that we may apply Plancherel's theorem and obtain its Fourier transform.

Consider first the special case when F(y) is continuous for $a \leq y \leq b$ and $F(y) = 0$ for $y < a$ and $y > b$. In this case, $H(x) = 0$ for $x < a$, while

$$(3.2) \quad H(x) = \begin{cases} e^{-x/2} \int_{a}^{x} F(y)e^{y/2}\,dy , & a \leq x \leq b \\ e^{-x/2} \int_{a}^{b} F(y)e^{y/2}\,dy , & x > b , \end{cases}$$

and since

$$H(x) = O(e^{-x/2}) , \quad \text{as } x \to \infty ,$$

we see that

$$H(x) \in L_2(-\infty,\infty) .$$

Differentiating (3.1), which is admissible, we get

$$H'(x) + \tfrac{1}{2} H(x) = F(x)$$

and since F and H belong to $L_2(-\infty,\infty)$, it follows that

$$H'(x) \in L_2(-\infty,\infty).$$

Hence, using theorem 61, we obtain

$$(3.3) \quad A(\alpha)\,(\tfrac{1}{2} - i\alpha) = \phi(\alpha)$$

where $A(\alpha)$ is the Fourier transform of H(x). Therefore,

$$\|H\| = \left(\int_{-\infty}^{\infty} |A(\alpha)|^2 \, d\alpha\right)^{\frac{1}{2}}$$

$$= \left(\int_{-\infty}^{\infty} \frac{|\phi(\alpha)|^2}{|\frac{1}{2} - i\alpha|^2} \, d\alpha\right)^{\frac{1}{2}}$$

$$\leqslant 2\left(\int_{-\infty}^{\infty} |\phi(\alpha)|^2 \, d\alpha\right)^{\frac{1}{2}} = 2\|F\| .$$

The transformation of F into H by the relation (3.2) is thus a bounded linear transformation on the set of continuous functions each vanishing outside an interval; this set is dense in L_2; hence the transformation has an abstract extension to the whole of L_2. That is, for every $F \in L_2(-\infty, \infty)$ we have

(3.4) $$H(x) = \underset{n\to\infty}{\text{l.i.m.}}\; e^{-x/2} \int_{-\infty}^{x} F_n(y) e^{y/2} \, dy, \quad H \in L_2 ,$$

where $F_n(y)$ is a sequence of continuous functions vanishing outside an interval and converging towards $F(y)$ in L_2-norm.

Now put

$$H_n(x) = e^{-x/2} \int_{-\infty}^{x} F_n(y) e^{y/2} \, dy ;$$

due to relation (3.1)', for each x, $H_n(x) \to H_o(x)$ where

(3.5) $$H_o(x) = e^{-x/2} \int_{-\infty}^{x} F(y) e^{y/2} \, dy .$$

Thus the sequence $\{H_n(x)\}$ converges in norm towards $H(x)$ and point-wise towards $H_o(x)$. Hence $H(x) = H_o(x)$ and thus (3.4) is identical with (3.1).

Collecting the results, we have

(3.6) $$H(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\alpha) e^{-iX\alpha}\, d\alpha ,$$

where

(3.7) $$A(\alpha) = \frac{\phi(\alpha)}{\frac{1}{2} - i\alpha} ,$$

and

(3.8) $$e^{-X/2} \int_{-\infty}^{X} F(y) e^{y/2}\, dy = H(X) = \int_{-\infty}^{\infty} K(y+X) G(y) dy$$

with the same notation as in §2.

Using (3.6) and (3.7) in (3.8), and applying Parseval's formula ((2.14),Chr.IV), we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iX\alpha} \frac{\phi(\alpha)}{\frac{1}{2} - i\alpha}\, d\alpha = \int_{-\infty}^{\infty} e^{-iX\alpha} R(\alpha)\psi(-\alpha) d\alpha .$$

Now, $\frac{\phi(\alpha)}{\frac{1}{2} - i\alpha}$ and $R(\alpha)\psi(-\alpha)$ belong to $L_1(-\infty,\infty)$, since they are products of functions belonging to L_2. Hence, by the uniqueness of Fourier transforms in L_1, we deduce that

$$\frac{1}{\sqrt{2\pi}} \frac{\phi(\alpha)}{\frac{1}{2} - i\alpha} = R(\alpha)\psi(-\alpha)$$

for almost all α. In other words,

(3.9) $$\phi(\alpha) = T(\alpha)\psi(-\alpha)$$

where

$$T(\alpha) = \sqrt{2\pi}\, R(\alpha) \left(\frac{1}{2} - i\alpha\right).$$

Similarly from the relation (2.3):

$$e^{-X/2} \int_{-\infty}^{\infty} G(y) e^{y/2}\, dy = \int_{-\infty}^{\infty} \overline{K(y+X)} F(y) dy$$

we obtain:

$$\frac{1}{\sqrt{2\pi}}\ \frac{\psi(\alpha)}{\frac{1}{2} - i\alpha} = \overline{R(-\alpha)}\phi(-\alpha),$$

or

(3.10) $$\psi(\alpha) = \overline{T(-\alpha)}\phi(-\alpha) .$$

Hence, from (3.9) and (3.10), we deduce that

$$\phi(\alpha) = T(\alpha) \cdot \overline{T(\alpha)}\phi(\alpha) ,$$

or,

(3.11) $$|T(\alpha)|^2 = 1 .$$

Thus we are led to the following

THEOREM 80. *To every unitary transformation in $L_2(0,\infty)$ representable by the Watson transform there corresponds a unitary transformation $\phi \to \psi$ of the form:*

(3.12)
$$U : \phi(\alpha) = T(\alpha)\psi(-\alpha)$$
$$U^{-1} : \psi(\alpha) = \overline{T(-\alpha)}\phi(-\alpha)$$

for $\phi(\alpha), \psi(\alpha) \in L_2(-\infty,\infty)$ with the condition,

(3.13) $$|T(\alpha)|^2 = 1$$

and the corresponding kernel $K(x)$ is real-valued if and only if

$$T(\alpha) = \overline{T(-\alpha)} .$$

Conversely, every transformation (3.12) derives from a certain Watson transform in the manner described.

Proof: The first part of the theorem was obtained in (3.9), (3.10) and (3.11). To prove the converse, we proceed as follows: given $T(\alpha)$ with property (3.13), consider two functions $\phi(\alpha)$, $\psi(\alpha) \in L_2(-\infty,\infty)$, and let their transforms be F and G respectively. Since $|T(\alpha)|^2 = 1$, it follows that

$$\frac{T(\alpha)}{\sqrt{2\pi}\,(\frac{1}{2} - i\alpha)} \in L_2(-\infty,\infty).$$

Therefore, there exists a function K(y), say, in $L_2(-\infty,\infty)$ such that

$$K(y) = \text{l.i.m.}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\alpha}\, \frac{T(\alpha)}{\sqrt{2\pi}\,(\frac{1}{2} - i\alpha)}\, d\alpha .$$

By Parseval's relation, we have therefore

$$(3.14)\quad \int_{-\infty}^{\infty} K(y+X)G(y)dy = \int_{-\infty}^{\infty} e^{-iX\alpha}\, \frac{T(\alpha)}{\sqrt{2\pi}\,(\frac{1}{2} - i\alpha)}\, \psi(-\alpha)d\alpha$$

Setting

$$\rho(t) = \begin{cases} e^{-t/2}, & 0 < t < \infty, \\ 0, & -\infty < t \leq 0 \end{cases}$$

and observing that

$$\int_0^{\infty} e^{-t/2 + i\alpha t}\, dt = \frac{1}{\frac{1}{2} - i\alpha}$$

we obtain

$$\int_{-\infty}^{X} F(y)e^{-\frac{X-y}{2}}\, dy = \int_{-\infty}^{\infty} F(y)\rho(X-y)dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iX\alpha}\, \frac{\phi(\alpha)}{\frac{1}{2} - i\alpha}\, d\alpha .$$

By hypothesis, $\phi(\alpha) = T(\alpha)\,\psi(-\alpha)$, and hence we have

$$(3.15)\quad e^{-X/2} \int_{-\infty}^{X} F(y)e^{y/2}\, dy = \int_{-\infty}^{\infty} K(y+X)G(y)dy$$

Similarly, we can also obtain:

$$(3.16)\qquad e^{-X/2}\int_{-\infty}^{X} G(y)e^{y/2}\,dy = \int_{-\infty}^{\infty}\overline{K(y+X)}\,F(y)dy\ .$$

By the definition of F and G we have

$$\int_{-\infty}^{\infty}|\phi|^2\,dx = \int_{-\infty}^{\infty}|F|^2\,dx$$

$$\int_{-\infty}^{\infty}|\psi|^2\,dx = \int_{-\infty}^{\infty}|G|^2\,dx$$

and, by hypothesis U is isometric, so that

$$\int_{-\infty}^{\infty}|\phi|^2\,dx = \int_{-\infty}^{\infty}|\psi|^2\,dx$$

hence

$$(3.17)\qquad \int_{-\infty}^{\infty}|F|^2\,dx = \int_{-\infty}^{\infty}|G|^2\,dx\ .$$

Further, since K(y) is the Fourier transform of $\frac{1}{\sqrt{2\pi}}\,\frac{T(\alpha)}{\frac{1}{2}-i\alpha}$, follows by Parseval's relation that

$$(3.18)\qquad \int_{-\infty}^{\infty}K(y+A)\overline{K(y+B)}dy = e^{-\left|\frac{A-B}{2}\right|}\ .$$

From (3.15) - (3.18) it follows that to the given unitary transformation there corresponds a Watson transform, and so theorem 80 is completely proved.

<u>Remark</u>: There are as many Watson transforms as there are functions of absolute value unity almost everywhere.

With the notation already in use, we have:

$$K(y) = \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} R(x)dx$$

$$R(x) = \frac{1}{\sqrt{2\pi}} \frac{T(x)}{\frac{1}{2} - ix} ,$$

$$K(y) = k(e^y)\, e^{-y/2} .$$

Hence,

$$(3.19) \qquad k(e^y)e^{-y/2} = \text{l.i.m.} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyx} \frac{T(x)}{\frac{1}{2} - ix} dx ,$$

or,

$$(3.20) \qquad \frac{k(x)}{x} = \text{l.i.m.} \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-\frac{1}{2} - i\alpha} \frac{T(\alpha)}{\frac{1}{2} - i\alpha} d\alpha , \ x > 0.$$

From (3.19) it follows that

$$(3.21) \qquad \frac{T(\alpha)}{\frac{1}{2} - i\alpha} = \text{l.i.m.} \int_{-\infty}^{\infty} k(e^y)e^{-y/2}\, e^{iy\alpha} dy$$

$$(3.22) \qquad = \text{l.i.m.} \int_{0}^{\infty} k(x)\, x^{i\alpha - 3/2} dx .$$

If we set

$$T(\alpha) = Q(\tfrac{1}{2} + i\alpha)$$

where $Q(t)$ is a complex function of the real variable t, then from (3.19) we get

$$(3.23) \qquad k(x) = \text{l.i.m.} \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{\frac{1}{2} - i\alpha} \frac{Q(\frac{1}{2} + i\alpha)}{\frac{1}{2} - i\alpha} d\alpha ,$$

$$(3.24) \qquad = \text{l.i.m.} \frac{1}{2\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{x^{1-s}}{1-s} Q(s)ds;$$

also purely formally we have

$$(3.25) \qquad k'(x) = \text{l.i.m.}\ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^{s}\, Q(s)ds\ .$$

We have seen that if $\phi \in L_2$, $\psi \in L_2$, then a necessary and sufficient condition for the transformation U defined by

$$U \ :\ \phi(\alpha) = T(\alpha)\,\psi(-\alpha)$$
$$U^{-1} :\ \psi(\alpha) = \overline{T(-\alpha)}\,\phi(-\alpha)$$

to be unitary is that

$$(3.26) \qquad |T(\alpha)|^2 = T(\alpha)\ \overline{T(\alpha)} = 1\ .$$

We have also seen that k(x) is real-valued if and only if

$$(3.27) \qquad T(\alpha) = \overline{T(-\alpha)}.$$

(3.26) and (3.27) together imply that

$$(3.28) \qquad T(\alpha)\,T(-\alpha) = 1\ .$$

Further, (3.28) and (3.27) imply (3.26); while, (3.28) and (3.26) imply (3.27). Now the conditions (3.26) and (3.28) can be expressed in terms of Q as follows:

$$(3.29) \qquad |Q(s)|^2 = 1, \quad \text{for } s = \tfrac{1}{2} + i\alpha$$

$$(3.30) \qquad Q(s) = \overline{Q(1-s)}, \text{ for } s = \tfrac{1}{2} + i\alpha$$

$$(3.31) \qquad Q(s)Q(1-s) = 1, \text{ for } s = \tfrac{1}{2} + i\ \ .$$

It follows that (3.29) and (3.30) together imply (3.31), while (3.31) and (3.30) (or (3.29)) imply (3.29) (or (3.30)). From theorem 80 we are thus led to the following

THEOREM 81. (i) *If* $k(x)$ *is real-valued, and* $\frac{k(x)}{x} \in L_2(0,\infty)$, *so that (by (3.22)) the integral*

$$\int_0^\infty k(x) x^{i\alpha - 3/2}\, dx$$

converges in mean to the function $\dfrac{Q(\frac{1}{2} + i\alpha)}{\frac{1}{2} - i\alpha} \in L_2(-\infty,\infty)$ *where* $Q(\frac{1}{2} + i\alpha)$ *is a complex-valued function of the real variable* α, *and satisfies (by (3.30)) the condition*

$$Q(\tfrac{1}{2} + i\alpha) = \overline{Q(\tfrac{1}{2} - i\alpha)} \quad ,$$

then the functional equation

$$Q(\tfrac{1}{2} + i\alpha)\, Q(\tfrac{1}{2} - i\alpha) = 1$$

must necessarily be satisfied in order that there may exist a unitary transformation $f = g$, $f \in L_2(0,\infty)$, *of the Watson type, namely,*

$$(3.22) \qquad \int_0^a g(x)dx = \int_0^\infty \frac{k(ay)}{y} f(x)dx$$

$$(3.33) \qquad \int_0^a f(x)dx = \int_0^\infty \frac{k(ay)}{y} g(y)dy$$

$$(3.34) \qquad \int_0^\infty |f|^2 dx = \int_0^\infty |g|^2\, dy \ .$$

(ii) *Conversely, if* $Q(\frac{1}{2} + i\alpha)$ *is a complex function of the real variable* α, $-\infty < \alpha < \infty$, *such that* $Q(\frac{1}{2} + i\alpha) = \overline{Q(\frac{1}{2} - i\alpha)}$, *and if, further*

$$Q(\tfrac{1}{2} + i\alpha)\, Q(\tfrac{1}{2} - i\alpha) = 1$$

then (by (3.20)) the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{T(\alpha)}{\frac{1}{2}-i\alpha}\,x^{-\frac{1}{2}-i\alpha}\,d\alpha\,,\quad x>0$$

converges in mean to a function $\frac{k(x)}{x}\in L_2(0,\infty)$ so that $\frac{k(ay)}{y}$ will be a kernel, satisfying the condition

$$\int_0^{\infty}\frac{k(ay)k(by)}{y}\,dy=\min(a,b)$$

for which (3.32) - (3.34) will hold.

Remarks: For real k(ay) we have $\tau=\tau^{-1}$ or $\tau^2=I$ (identity). Now for any element f we can put uniquely

$$f=f_+ + f_-$$

where $\tau f_+ = f_+$, $\tau f_- = -f_-$, so that

$$f_+=\frac{f+\tau f}{2}\,,\quad f_-=\frac{f-\tau f}{2}\,.$$

Thus, for k real, every $f\in L_2$ is the unique sum of a self-reciprocal function with a skew-reciprocal function. In terms of our Fourier transforms $\phi(\alpha)$, $\psi(\alpha)$ for a self-reciprocal function we have

$$\phi_+(\alpha)=T(\alpha)\,\phi_+(-\alpha)$$

or

$$\phi_+(-\alpha)=T(-\alpha)\,\phi_+(\alpha)$$

and for the skew-reciprocal function we have

$$\phi_-(\alpha)=-T(-\alpha)\,\phi_-(\alpha)\,.$$

Thus all self-reciprocal functions arise by taking for $\phi_+(\alpha)$ in $0 \leq \alpha < \infty$ any function $\phi(\alpha) \in L_2(0,\infty)$ and then putting $\phi(-\alpha) = T(-\alpha)\,\phi(\alpha)$. Similarly for skew-reciprocal functions, and the argument shows that "there are as many functions of the one kind as of the other."

Take a general (non-real) Watson-transform on $-\infty < \alpha < \infty$. We have

$$\psi(\alpha) = T(\alpha)\,\phi(-\alpha).$$

For this, we write

$$\psi_1(\alpha) = \phi(-\alpha) \tag{3.35}$$

$$\begin{cases} \psi(\alpha) = T(\alpha)\,\psi_1(\alpha) \quad \text{with} \\ |T(\alpha)| = 1 \end{cases} \tag{3.36}$$

Now (3.35) is a Watson transform with $T(\alpha) = 1$, and this is, in $0 < x < \infty$

$$g(x) = \frac{1}{x} f\left(\frac{1}{x}\right).$$

We call this <u>the</u> elementary Watson transform. Then, by theorem 80 we have the following

<u>THEOREM 82</u>. <u>In $L_2(0,\infty)$ any Watson transform is the product of the elementary Watson transform with a non-Watson transform which on $L_2(-\infty < X < \infty)$, is a unitary operator commutative with translations</u>.

CHAPTER VI

GENERAL TAUBERIAN THEOREMS

§1. Introduction

In the Tauberian theorems of Chapter I, we started from a "kernel" $K(\alpha)$, formed its linear combinations

$$\Lambda(\alpha) = \sum_{m=1}^{n} c_m K(\alpha \lambda_m) \tag{1.1}$$

and then considered the extension $E\{K(\alpha)\}$ which was obtained from (1.1) by the Riemann-Darboux process of inclusion from above and below. In the actual analysis we restricted ourselves to the kernel $K(\alpha) = e^{-\alpha}$; in this case, $E\{K(\alpha)\}$ consisted of a certain vector-space of functions in $0 \leq \alpha < \infty$ which contained, in particular, the function

$$F(\alpha) = \begin{cases} 1, & 0 \leq \alpha < 1, \\ 0, & 1 \leq \alpha < \infty. \end{cases} \tag{1.2}$$

Although the function (1.2) is a very special function, yet the application of the closure process gives rise to functions which are rather general in a way. In fact, we have the following proposition:

The Karamata extension $E\{F\}$ of $F(\alpha)$ as given by (1.2) consists precisely of functions $G(\alpha)$ each of which vanishes outside some finite interval (depending on the choice of $G(\alpha)$), and is Riemann integrable in any interval outside of which it vanishes.

For, the linear combinations $\Lambda(\alpha)$ are precisely step-functions (each vanishing outside a finite interval); each element of the extension, being "included" by two such step-functions from above and below, must itself therefore vanish outside a finite interval; but on every finite interval, due to the inclusion from above and below, it must be Riemann integrable; therefore, the elements of $E\{F\}$ must have the stated property; conversely, any function which is Riemann integrable and vanishes outside a finite interval can be so included.

Now suppose we want to replace, in a Tauberian theorem, the kernel $e^{-\alpha}$ by some other kernel $K(\alpha)$ with a view to obtaining the same summability as a conclusion. This then would demand that $E\{K\}$ contain $F(\alpha)$ as given by (1.2), and hence also contain $E\{F\}$ which, as we have seen, includes quite a large class of "arbitrary" functions. Replacing in (1.1), α by e^x and λ by e^{-y}, $-\infty < x < \infty$, $-\infty < y < \infty$, it follows that the necessary condition is that a suitable closure of linear transformations

$$(1.3) \qquad \sum c_n f(x-y_n), \quad K(e^x) = f(x)$$

shall contain many "arbitrary" functions $g(x)$ in $-\infty < x < \infty$. If we agree to replace in (1.3) the finite sum by an integral, thus writing

$$g(x) = \int_{-\infty}^{\infty} f(x-y)h(y)dy \ , \tag{1.4}$$

then the problem poses itself in the following way. Under what conditions on f(x) will formula (1.4), for a suitable choice of h(y), represent an "arbitrary" function g(x). Introducing, purely formally, Fourier transforms (of class L_1 or L_2) we see that (1.4) is equivalent with

$$\psi(t) = \phi(t)\chi(t) \tag{1.5}$$

where ϕ, ψ, χ are the transforms of f,g,h respectively. Since $\psi(t)$ is "arbitrary" it may be $\neq 0$ for any fixed t; therefore $\phi(t)$ must be $\neq 0$. Thus the only functions f(x) likely to satisfy the requirements are those whose Fourier transform nowhere vanishes. Thus

$$\int_{-\infty}^{\infty} f(x)e^{ixt}\, dx \neq 0 \ ,$$

is a necessary condition; or, going back to $\alpha = e^x$, we obtain the heuristic conclusion that only those kernels for which

$$\int_0^{\infty} K(\alpha)\alpha^{it-1}\, d\alpha \ \neq 0$$

everywhere, would be eligible. For $K(\alpha) = e^{-\alpha}$ this amounts to $\Gamma(it) \neq 0$ which indeed is the case. Now, the remarkable thing is that this one condition conversely is also sufficient to produce the expected consequences. Thus, more or less, the property of $\Gamma(it)$ actually used in the classical Tauberian theorem (that Abel summability implies Cesaro summability) is only its non-vanishing and no other specifiable property.

From the above heuristic considerations it becomes clear that the essence of the proof of the Tauberian theorem 30 of Chapter I lies in the study of the closure of a set of translations of a given function, namely $f(x + \log \lambda_n)$ as given in (1.3), and this, in turn, is bound up with the zeros of the Fourier transform of $f(x)$. In the case of the Karamata-extension, the closure is a Riemann - Darboux closure of translations of a Riemann integrable function. If we consider the set of all translations of more general functions, say in L_1 or in L_2, and the corresponding Lebesgue-closure (that is, in the sense of the norm) one might expect to obtain more general Tauberian theorems. In fact, such a closure-theorem for L_2 has been proved already in Chapter IV. Its L_1 analogue has been used by Wiener to prove a general Tauberian theorem which is most closely identified with his method of employing the Fourier transform. However the resulting version of the Tauberian theorem is only one of the many possible, and in the proof of the others, some of which are of considerable analytic interest, the underlying closure-property must be worked in with other syllogistic devices. In what follows, several such versions will be elaborated, and we will effect a combination of general kernels under the integral sign with general averaging processes of "slow growth" under the limit-sign.

§2. Preliminary lemmas

Definition 1. The class Ψ_p , $p = 1,2,3,\ldots$ is defined to include all functions $\psi(x)$ in $-\infty < x < \infty$ with the following three properties:

(2.1) $\psi(x)$ is positive, continuous and has the value 1 in $-\infty < x < 0$;

(2.2) for any finite real numbers $a_0 < a_1$, we have

(2.2)' $$\lim_{x\to\infty} \frac{\psi(x+a)}{\psi(x)} = 1$$

uniformly in (a_0,a_1), and there exist finite real numbers g_0,g_1 depending on a_0,a_1 such that

(2.2)" $$g_0 < \frac{\psi(x+a)}{\psi(x)} < g_1 ;$$

for $a_0 < a < a_1$;

(2.3) there is a constant $c = c_\psi$ such that

$$\sum_{n=-\infty}^{\infty} \frac{\psi(n)}{1+|x-n|^{p+1}} \leq c_1 \psi(x), \quad -\infty < x < \infty.$$

Or, alternatively,

(2.3)' $$\int_{-\infty}^{\infty} \frac{\psi(y)dy}{1+|x-y|^{p+1}} \leq c_1 \psi(x)$$

where c_1 (like c_2, c_3, etc.) is a constant.

Note that if (2.1) and (2.2)' are given, then (2.3) and (2.3)' are indeed equivalent.

The class of functions Ψ_p may be termed "functions of slow growth", the standard of comparison being the exponential function e^x as $x \to \infty$. It will be shown presently that the typical function

$$x^{\sigma_1}(\log x)^{\sigma_2}(\log\log x)^{\sigma_3}\ldots$$

will comply with our requirements. We need the following

<u>Lemma 1</u>. Let $\psi(x)$ satisfy conditions (2.1) and (2.2). If there are real numbers ρ, σ with $\rho \geq 0$ such that

$$\psi(x) = O(x^{\rho}) \ , \quad \frac{1}{\psi(x)} = O(\frac{1}{x^{\sigma}}) \ , \quad \text{as } x \to \infty$$

then (2.3)' [and hence (2.3)] is fulfilled for

$$\text{(2.4)} \qquad p \geq \max\,(1+\rho, \rho-\sigma) \ .$$

Proof. It will be sufficient to show that the integral

$$\int_{-\infty}^{\infty} \frac{\psi(y)dy}{1+|x-y|^{p+1}}$$

is (a) bounded for $x \leq 0$, and (b) $O(\psi(x))$ as $x \to \infty$. As regards (a) we have:

$$\int_{-\infty}^{1} \leq c_1 \int_{-\infty}^{1} (1+|x-y|^{p+1})^{-1}\, dy \leq c_2 \ ,$$

$$\int_{1}^{\infty} \leq c_3 \int_{1}^{\infty} y^{-p-1}\, y^{\rho}\, dy = c_4 \ .$$

As regards (b) we have:

$$\int_{-\infty}^{1} = O\left(\int_{-1}^{\infty} (x+y)^{-p-1} dy\right) = O(x^{-p}) = O\{x^{-p-\sigma}\psi(x)\} = O(\psi(x))$$

$$\int_{1}^{x/2} = O\left(\int_{1}^{x/2} (x-y)^{-p-1} y^{\rho} dy\right) = O(x^{-p+\rho}) = O\{x^{-p+\rho-\sigma}\psi(x)\}$$

$$= O(\psi(x))$$

$$\int_{x/2}^{2x} = O(\psi(x)) \int_{-\infty}^{\infty} (1+|x-y|^{p+1})^{-1} dy = O\{\psi(x)\}$$

$$\int_{2x}^{\infty} = O\left(\int_{2x}^{\infty} y^{-p-1} y^{\rho} dy\right) = O(x^{-p+\rho}) = O(\psi(x)) .$$

The proof of the lemma is now immediate.

Assume, for instance, that for large x,

$$\psi(x) = x^{\sigma_1} (\log x)^{\sigma_2} (\log \log x)^{\sigma_3} \ldots (\log\ldots\log x)^{\sigma_n}$$

Then, for $\sigma_1 > 0$ we may put $\rho = \sigma_1 + \varepsilon$, $\sigma = \sigma_1 - \varepsilon$ for any fixed $\varepsilon > 0$, and (2.3) will be satisfied for

$$p > \max (1 + \sigma_1 + \varepsilon , 2\varepsilon) .$$

Or,

$$p > 1 + \sigma_1 .$$

If, however, $\sigma_1 < 0$, we have to put

$$\rho = \varepsilon , \quad \sigma = \sigma_1 - \varepsilon$$

$$p > \max (1 + \varepsilon , 2\varepsilon - \sigma_1)$$

$$p > \max (1, -\sigma_1) = \max (1, |\sigma_1|) .$$

For $\sigma_1 = \sigma_2 = \ldots = \sigma_n = 0$, we have $\psi(x) = 1$ and $p = 1$.

Definition 2. The class K_q, $q = 0,1,2,\ldots$ is defined to include all continuous functions in $-\infty < x < \infty$ which are $O(|x|^{-q})$ as $|x| \to \infty$.

Lemma 2. If a function $F(\alpha)$ has q continuous derivatives in $-\infty < \alpha < \infty$, and if the functions $F(\alpha)$, $F^{(1)}(\alpha)$, ..., $F^{(q)}(\alpha)$ are absolutely integrable in $-\infty < \alpha < \infty$, then the function

$$f(x) = \int_{-\infty}^{\infty} F(\alpha) e^{ix\alpha}\, d\alpha$$

belongs to K_q.

Proof. By partial integration.

Lemma 3. For $q \geq 1$, if $f(x)$ belongs to K_{q+1} then its transform

$$F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx$$

has q-1 derivatives; and the $(q-1)^{th}$ derivative $F^{(q-1)}(\alpha)$ is locally absolutely continuous, and $F^{(q)}(\alpha)$ belongs to $L_2(-\infty,\infty)$.

Proof. Follows from theorem 65, since

$$(ix)^q f(x) \in L_2(-\infty,\infty) .$$

Lemma 4. If $S(x)$ is of bounded variation in every finite interval, and $\psi(x) \in \Psi_p$, and

$$\int_x^{x+1} |dS(y)| \leq c_1\, \psi(x), \quad -\infty < x < \infty ,$$

then there exists a constant c such that

$$|f(x)| \leq c_2\, (1 + |x|^{p+1})^{-1}$$

implies

$$\int_{-\infty}^{\infty} |f(x-y)| \; |dS(y)| \leq c \, . \, c_2 \, \psi(x) \, .$$

Proof: By application of (2.3) in conjunction with (2.2).

Lemma 5. There is a number c, depending on q, such that

$$\int_{-\infty}^{\infty} (1 + |x-y|^{q+2})^{-1} (1 + |y|^{q+2})^{-1} dy \leq c \, (1 + |x|^{q+2})^{-1} \, .$$

Proof: By decomposition of the range of integration into two parts, which, for instance, for positive x are $(-\infty, \frac{1}{2}x)$, $(\frac{1}{2}x, \infty)$.

Lemma 6. If f(x) and h(x) are functions of K_{q+2} and

$$\int_{-\infty}^{\infty} h(y) dy = 1$$

and if we put

$$f_n(x) = \int_{-\infty}^{\infty} f(x + \frac{y}{n}) h(y) dy = n \int_{-\infty}^{\infty} f(x+y) h(ny) dy$$

then

$$|f(x) - f_n(x)| \leq \varepsilon_n \, (1 + |x|^{q+1})^{-1}$$

where

$$\lim_{n \to \infty} \varepsilon_n = 0.$$

Proof: Since

$$|f_n(x) - f(x)| = | \int_{-\infty}^{\infty} (f(x + \frac{y}{n}) - f(x)) h(y) dy |$$

$$\leq \int_{-\delta}^{\delta} |f(x + \frac{y}{n}) - f(x)| \; |h(y)| dy + \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} ,$$

it follows that $f_n(x) \to f(x)$ uniformly in every finite x-interval. The lemma now results from the fact that

$$f_n(x) = O(x^{-q-2}) , \quad |x| \to \infty ,$$

uniformly for $n \geq 1$. This may be shown (supposing, for example, that x is positive) by splitting the range of integration in

$$|f_n(x)| \leq c.n \int_{-\infty}^{\infty} (1 + |x+y|^{q+2})^{-1} (1+n^{q+2}|y|^{q+2})^{-1} dy$$

into the intervals $(-\infty, -\frac{1}{2} x), (-\frac{1}{2} x, \infty)$.

<u>Definition 3.</u> Given functions $S(x)$ and $\psi(x)$ as in Lemma 4, and a constant A, we say that a function $f(x)$ of the class K_{p+1} is a W_p-function if the function defined by

$$T_f(x) = \frac{1}{\psi(x)} \int_{-\infty}^{\infty} f(x-y) dS(y)$$

which, by Lemma 4, is bounded, tends for $x \to \infty$, to the limit

$$A \int_{-\infty}^{\infty} f(y) dy .$$

<u>Lemma 7.</u> If $g_1(x), g_2(x), \ldots$ are W_p-functions, and if there is a function $g(x)$ such that

$$|g(x) - g_n(x)| \leq \varepsilon_n (1+|x|^{p+1})^{-1}, \quad \varepsilon_n \to 0 ,$$

then $g(x)$ is also a W_p-function.

Proof: By lemma 5.

Lemma 8. If $g(x)$ is a W_p-function, and $r(x)$ is an arbitrary function of K_{p+2}, then the convolution

$$h(x) = \int_{-\infty}^{\infty} r(z)g(x-z)dz$$

is also a W_p-function.

Proof: We first assume that $r(z)$ vanishes outside an interval (z_o, z_1). The function

$$T_g(x,z) = \frac{1}{\psi(x)} \int_{-\infty}^{\infty} g(x-z-y)dS(y) \tag{2.5}$$

has the value

$$\frac{\psi(x-z)}{\psi(x)} \ T_g(x-z) \, .$$

On account of (2.2)' it converges, for $x \to \infty$, to

$$A \int_{-\infty}^{\infty} g(y-z)dy = A \int_{-\infty}^{\infty} g(y)dy$$

uniformly in $z_o < z < z_1$. If we multiply (2.5) by $r(z)$ and integrate between z_o and z_1, we obtain the desired result, since

$$\int_{-\infty}^{\infty} r(z)dz \int_{-\infty}^{\infty} g(y)dy = \int_{-\infty}^{\infty} h(y)dy \, .$$

In the general case, there are smoothing functions $r_n(x)$ vanishing outside finite intervals such that

$$|r(x)-r_n(x)| \leq \varepsilon_n \, (1+|x|^{p+1})^{-1} , \quad \varepsilon_n \to 0 \, .$$

Also if $h_n(x)$ is the convolution of $r_n(x)$ with $g(x)$, we have

$$|h(x) - h_n(x)| \leq \delta_n \, (1+|x|^{p+1})^{-1}, \quad \delta_n \to 0 ,$$

by lemma 6; and therefore lemma 7 applies.

§3. Tauberian theorems; averages on $(-\infty, \infty)$

THEOREM 83. Assumptions:

(i) $\psi(x) \in \overline{\Psi}_p$

(ii) $S(x)$ is of bounded variation in every finite interval, and

$$\int_x^{x+1} |dS(y)| \leq c_1 \, \psi(x), \quad -\infty < x < \infty$$

(iii) $g(x)$ is a function of class K_{p+3} (hence also of class K_{p+1}) and

$$\frac{1}{\psi(x)} \int_{-\infty}^{\infty} g(x-y)dS(y)$$

exists for every x, and is bounded in $-\infty < x < \infty$

(iv) $$G(\alpha) = \int_{-\infty}^{\infty} g(x)e^{-i\alpha x} \, dx \neq 0 , \quad -\infty < \alpha < \infty,$$

(v) there is an A such that

$$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} g(x-y)dS(y) = A \int_{-\infty}^{\infty} g(y)dy .$$

Conclusion: for any $f(x) \in K_{p+2}$, we have

$$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} f(x-y)dS(y) = A \int_{-\infty}^{\infty} f(y)dy$$

Proof: Let $H(\alpha)$ be any function having p+2 continuous derivatives, and vanishing outside a finite interval. By lemma 3, the function $G(\alpha)$ has p+1 continuous derivatives and a $(p+2)^{th}$ derivative which is in L_2 for $-\infty < \alpha < \infty$ and hence is in L_1 in every finite interval. Since $G(\alpha) \neq 0$ everywhere by assumption (iv), it follows that the quotient

$$\frac{H(\alpha)}{G(\alpha)}$$

has p+2 derivatives belonging to L_1. Hence by Lemma 2, the function

$$r(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{H(\alpha)}{G(\alpha)} e^{i\alpha x} d\alpha$$

is a function of K_{p+2}. But the function

$$h(x) = \int_{-\infty}^{\infty} H(\alpha)\, e^{i\alpha x}\, d\alpha$$

is the convolution of $r(x)$, $g(x)$ and therefore by Lemma 8 it is a W_{p+1}-function. The same holds true for the function $h(-nx)$. Now the function

$$f_n(x) = \int_{-\infty}^{\infty} f(x + \frac{y}{n})h(y)dy$$

being the convolution of $f(x)$ and $n\,h(-nx)$ is also a W_{p+1}-function. If we suppose that $H(0) = 1$, then our final assertion about $f(x)$, namely, that $f(x) \in W_{p+1}$ follows from Lemmas 6 and 7.

<u>THEOREM 84</u>. <u>Assumptions</u>:

(i) $\psi(x) \in \mathfrak{T}_p$

(ii) S(x) is real and of bounded variation in every finite interval, and

$$\int_x^{x+1} |dS(y)| - \int_x^{x+1} dS(y) \leq c_1\psi(x)$$

(this is a "one-sided" assumption and is automatically fulfilled if S(x) is monotonic)

(iii) $g(x) \geq 0$ and $g(x) \in K_{p+3}$, and

(iv) $G(\alpha) \equiv \int_{-\infty}^{\infty} g(x)e^{-i\alpha x}\, dx \neq 0$, $-\infty < \alpha < \infty$,

(v) for each x, the integral

$$\int_{-\infty}^{\infty} g(x-y)dS(y)$$

exists as a Cauchy limit: $\lim_{T\to 0} \int_{-T}^{T}$ (or, as some other suitable limit, the same for all x), and

$$\left|\int_{-\infty}^{\infty} g(x-y)dS(y)\right| \leq c_2\, \psi(x)$$

(vi) there exists a number A such that

$$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} g(x-y)dS(y) = A \int_{-\infty}^{\infty} g(y)dy$$

Conclusion: we have

(vii) $$\int_x^{x+1} |dS(y)| \leq c_4\, \psi(x)$$

and hence, by theorem 83, we have for $f(x) \in K_{p+2}$,

(viii) $$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} f(x-y)dS(y) = A \int_{-\infty}^{\infty} f(y)dy \, .$$

Proof: Define the monotone function $\rho(x)$ by the relation

$$\rho(x) = \int_0^x |dS(y)| - \int_0^x dS(y)$$

for all real x in $(-\infty,\infty)$. We have

$$\int_{-\infty}^{\infty} g(x-y)|dS(y)| = \int_{-\infty}^{\infty} g(x-y)d\rho(y) + \int_{-\infty}^{\infty} g(x-y)dS(y)$$

$$(3.1) \qquad = I_1 + I_2 \text{ , say.}$$

In (3.1) we note that I_2 exists by explicit assumption (v) while I_1 exists by assumption (ii)of the theorem in conjunction with $g(x) \in K_{p+3}$. Also,

$$|I_2| \leq c_2\,\psi(x) \text{ , } |I_1| \leq c_4\,\psi(x) \text{ ,}$$

and thus

$$\int_{-\infty}^{\infty} g(x-y)|dS(y)| \leq c\,\psi(x) \text{ .}$$

Now put

$$\sigma(x) = \int_0^x |dS(y)| \text{ ;}$$

then

$$\int_{-\infty}^{\infty} g(x-y)d\sigma(y) \leq c\,\psi(x).$$

Since $G(\alpha) \neq 0$, $g(x)$ is not identically zero. Suppose that k is an integer and that $g(x) \geq c_1 > 0$ for $a \leq x \leq a + \frac{1}{k}$. Now

$$\int_{a\leq x-y\leq a+\frac{1}{k}} g(x-y)d\sigma(y) \leq c\,\psi(x)$$

and the left side is

$$\sum c_1 \int_{a \leq x-y \leq a+\frac{1}{k}} d\sigma(y)$$

so that

$$\sigma(x-a) - \sigma(x-a-\tfrac{1}{k}) \leq c\,\psi(x)$$

or,

$$\sigma(x+\tfrac{1}{k}) - \sigma(x) \leq c\,\psi(x+a+\tfrac{1}{k})$$

or

$$\sigma(x+1) - \sigma(x) \leq c \sum_{j=1}^{k} \psi(x+a+\tfrac{1}{k}) .$$

However if we use property (2.2) we finally obtain

$$\sigma(x+1) - \sigma(x) \leq c\,\psi(x) ,$$

which implies the conclusion of our theorem.

THEOREM 85. Assumptions:

(i) $\psi(x) \in \Phi_p$

(ii) $S(x)$ is real and of bounded variation in every finite interval, and

$$\int_x^{x+1} |dS(y)| - \int_x^{x+1} dS(y) \leq c_1\,\psi(x)$$

(iii) corresponding to each $c > 0$, there exists a $g_c(x) \geq 0$ with $g_c(x) \in K_{p+3}$ and

$$G_c(\alpha) = \int_{-\infty}^{\infty} g_c(x) e^{-i\alpha x}\,dx \neq 0, -\frac{2\pi}{c} < \alpha < \frac{2\pi}{c} ,$$

(iv) for all c, the integral

(iv, a_1) $$\frac{1}{\psi(x)} \int_{-\infty}^{\infty} g_c(x-y)\,dS(y)$$

<u>exists as a Cauchy limit, and for one particular</u> $c = c_o$ <u>we have</u>

$$(\text{iv}, a_2) \qquad \left|\frac{1}{\psi(x)} \int_{-\infty}^{\infty} g_{c_o}(x-y)dS(y)\right| \leq c_2$$

(v) <u>there exists a number</u> A <u>such that for all</u> c,

$$(\text{v}, a_1) \quad \lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} g_c(x-y)dS(y) = A \int_{-\infty}^{\infty} g_c(y)dy$$

<u>Conclusion</u>: <u>for</u> $f(x) \in K_{p+2}$, <u>we have</u>

$$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} f(x-y)dS(y) = A \int_{-\infty}^{\infty} f(y)dy \ .$$

Proof: In fact, by using $g_{c_o}(x)$ we prove as before that

$$\sigma(x+1) - \sigma(x) \leq c\,\psi(x) \ .$$

Now, if $H(\alpha)$ vanishes outside a closed finite interval, then choose some c such that this closed interval is contained in $-\frac{2\pi}{c} < \alpha < \frac{2\pi}{c}$. For this c we can form as before

$$r_c(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H(\alpha)}{G_c(\alpha)}\, e^{i\alpha x}\, d\alpha$$

and it has all the properties stated in the proof of theorem 83. In particular, the function

$$h(x) = \int_{-\infty}^{\infty} H(\alpha)e^{i\alpha x}\, d\alpha$$

is a W_{p+1} function. Now the proof is to be finished as in theorem 83.

Remarks: In the above three theorems we have proved that if the relation,

$$(3.2) \quad \lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} g(x-y)dS(y) = A \int_{-\infty}^{\infty} g(y)dy$$

holds for a special function g, then the relation

$$(3.3) \quad \lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} f(x-y)dS(y) = A \int_{-\infty}^{\infty} f(y)dy ,$$

holds for any general function $f \in K_{p+2}$. However, in a group of theorems following now, we shall deal with the case when the right side of (3.2) is simply equal to A, and our conclusion will undergo a corresponding modification.

THEOREM 86. Assumptions:

(i) $\psi(x) \in \Psi_p$

(ii) S(x) is real and of bounded variation in every finite interval, and

$$\int_x^{x+1} |dS(y)| - \int_x^{x+1} dS(y) < c\,\psi(x)$$

(iii) the function P(x) is monotonely non-decreasing in $-\infty < x < \infty$ and,

$$P(x) = 1 + e^{-ax} , \quad \text{as } x\to\infty$$

and

$$P(x) = e^{ax} , \quad \text{as } x\to -\infty$$

for some fixed $a > 0$.

(iv) for $0 < \varepsilon < a$, we can thus form the absolutely convergent integral

$$R(\varepsilon+i\alpha) = \int_{-\infty}^{\infty} P(x)e^{-\varepsilon x-i\alpha x}\,dx, \quad -\infty < \alpha < \infty,$$

and the assumption is that

$$\lim_{\varepsilon\downarrow+0} R(\varepsilon+i\alpha)$$

which can be proved to exist in $-\infty < \alpha < 0$ and $0 < \alpha < \infty$ is different from zero on these two half-lines.

(v) $$\frac{1}{\psi(x)} \int_{-\infty}^{\infty} P(x-y)dS(y)$$

exists, perhaps only as a Cauchy limit, and is bounded in $-\infty < x < \infty$

(vi) for some A

$$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} P(x-y)dS(y) = A .$$

Conclusion: (a) we have again

(3.4) $$\int_{x}^{x+1} |dS(y)| \leq c\,\psi(x)$$

(b) for any $f(x) \in K_{p+2}$ we have

(3.5) $$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} f(x-y)dS(y) = 0 ,$$

and hence

(c) if $Q(x)$ is a function such that

$$Q(x) \to 1 \quad \text{as } x \to \infty,$$
$$Q(x) \to 0 \quad \text{as } x \to -\infty,$$

and

(3.6) $$P(x) - Q(x) \in K_{p+2} ,$$

then

(3.7) $$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} Q(x-y)dS(y) = A.$$

Proof: For every $c > 0$, we introduce the function

(3.8) $$g_c(x) = P(x) - P(x-c)$$

and we will show that it has properties (iii), (iv) and (v) of the previous theorem 85, the last one with $A = 0$. This then proves conclusions (a) and (b) by theorem 85, and therefore also (c).

In fact, due to assumption (iii) on $P(x)$ we certainly have $g_c(x) \in K_{p+3}$ and $g_c(x) \geq 0$. Next, for $0 < \varepsilon < \alpha$ we have

$$R(\varepsilon+i\alpha) = \int_{-\infty}^{0} P(x)e^{-\varepsilon x-i\alpha x}\,dx + \int_{0}^{\infty} (P(x)-1)e^{-\varepsilon x-ix\alpha}\,dx + \int_{0}^{\infty} e^{-\varepsilon x-ix\alpha}\,dx .$$

Since

$$\int_{0}^{\infty} e^{-\varepsilon x-ix\alpha}\,dx = \frac{1}{\varepsilon+i\alpha} ,$$

we obtain

$$R(0+i\alpha) = \int_{-\infty}^{0} P(x)e^{-ix\alpha}\,dx + \int_{0}^{\infty} (P(x)-1)e^{-i\alpha x}\,dx + \frac{1}{i\alpha}$$

for $\alpha \neq 0$, the integrals existing. Also, for $\alpha = 0$, we obtain the existence of the limit-relation

$$(3.9) \qquad \lim_{\varepsilon \to 0} \varepsilon R(\varepsilon) = 1 .$$

Now, for the Fourier transform

$$G_c(\alpha) = \int_{-\infty}^{\infty} g_c(x) e^{-i\alpha x} dx$$

we can easily obtain

$$G_c(\alpha) = \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} g_c(x) e^{-(\varepsilon x + i\alpha x)} dx$$

for all α in $-\infty < \alpha < \infty$. But for fixed $\varepsilon > 0$ we have

$$\int_{-\infty}^{\infty} g_c(x) e^{-(\varepsilon + i\alpha)x} dx = \int_{-\infty}^{\infty} P(x) e^{-(\varepsilon + i\alpha)x} dx - \int P(x-c) e^{-(\varepsilon+i\alpha)x} dx$$

$$= R(\varepsilon + i\alpha)(1 - e^{-(\varepsilon + i\alpha)c}) .$$

Now, for $\alpha \neq 0$, the limit of this as $\varepsilon \to 0$ is

$$(3.10) \qquad R(i\alpha)\ (1 - e^{-i\alpha c})$$

which is $\neq 0$ for $0 < |\alpha| < \frac{2\pi}{c}$; and for $\alpha = 0$ it is

$$R(\varepsilon)\ (1 - e^{-\varepsilon c})$$

and due to (3.9) its limit as $\varepsilon \to 0$ is

$$c \lim_{\varepsilon \to 0} \varepsilon R(\varepsilon) = c > 0 .$$

Thus altogether we have verified assumption (iii) of theorem 85. The existence of (iv, a_1) of theorem 85 as a Cauchy limit, and (iv, a_2) for every c, follow now from the present assumption (v). Thus we have only to show that

(3.11) $$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} g_c(x-y)dS(y) = 0$$

in order to justify all our conclusions. Now,

(3.12) $$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} P(x-y)dS(y) = A$$

by assumption, hence, for fixed $c > 0$, also

$$\frac{1}{\psi(x-c)} \int_{-\infty}^{\infty} P(x-c-y)dS(y) \to A;$$

but

$$\frac{\psi(x-c)}{\psi(x)} \to 1, \quad \text{as } x \to \infty,$$

and hence

(3.13) $$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_{-\infty}^{\infty} P(x-c-y)dS(y) = A$$

and (3.11) follows by subtracting (3.13) from (3.12). This finishes the proof of our theorem.

Put now

$$Q(y) = \begin{cases} 0 & , -\infty < y < -\varepsilon \\ 1+y/\varepsilon & , -\varepsilon \leq y \leq 0 \\ 1 & , 0 < y < \infty \end{cases}$$

This function Q(y) satisfies all assumptions of our previous theorem for any p, and thus we have

<u>THEOREM 87</u>. <u>Under the assumptions of theorem 86</u>, <u>we have</u>:

(3.14) $$\lim_{x\to\infty} \frac{1}{\psi(x)} \left[\int_{-\infty}^{x} dS(y) + \int_{x}^{x+\varepsilon} \left(1 + \frac{x-y}{\varepsilon}\right)dS(y)\right] = A .$$

From this we will conclude the following

THEOREM 88. *If to the assumptions of theorem 86 we add the assumption that $S(y)\to 0$ as $y\to-\infty$ (for instance, $S(y) = 0$ in a half-line $-\infty < y < y_0$), then for every $\varepsilon > 0$ we have*

$$(3.15) \qquad \lim_{x\to\infty} \frac{1}{\psi(x)} \frac{1}{\varepsilon}\int_x^{x+\varepsilon} S(y)dy = A .$$

Proof: Under the additional assumption on $S(x)$ we have

$$\int_{-\infty}^{x} dS(y) = S(x)$$

and

$$\int_x^{x+\varepsilon} (1+\frac{x-y}{\varepsilon})dS(y) = -S(x) + \frac{1}{\varepsilon}\int_x^{x+\varepsilon} S(y)dy ,$$

and if we substitute this in (3.14) we obtain (3.15).

Now define the function $M_\varepsilon(x)$ by the relation

$$(3.16) \qquad M_\varepsilon(x) = \frac{1}{\psi(x)\varepsilon}\int_x^{x+\varepsilon} S(y)dy .$$

Then,

$$(3.17) \qquad \frac{S(x+\varepsilon)}{\psi(x)} = \frac{1}{\psi(x)\varepsilon}\int_x^{x+\varepsilon} [S(x+\varepsilon)-S(y)]dy + M_\varepsilon(x)$$

and

$$(3.18) \qquad M_\varepsilon(x) = \frac{1}{\psi(x)\varepsilon}\int_x^{x+\varepsilon} [S(y)-S(x)]dy + \frac{S(x)}{\psi(x)} .$$

We have proved that

$$(3.19) \qquad \lim_{x\to\infty} M_\varepsilon(x) = A .$$

Then, since

$$\overline{\lim}\,(U + V) \geq \underline{\lim}\, U + \overline{\lim}\, V$$

we obtain from (3.17) and (3.18):

$$\varliminf_{x\to\infty} \frac{S(x+\varepsilon)}{\psi(x)} = \varliminf \frac{S(x+\varepsilon)}{\psi(x+\varepsilon)} = \varliminf \frac{S(x)}{\psi(x)}$$

$$\geq \varliminf_{x\to\infty} \frac{1}{\psi(x)} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} [S(x+\varepsilon)-S(y)]dy + A \,, \tag{3.20}$$

and

$$A \geq \varliminf_{x\to\infty} \frac{1}{\psi(x)\varepsilon} \int_x^{x+\varepsilon} [S(y)-S(x)]dy + \varlimsup_{x\to\infty} \frac{S(x)}{\psi(x)} \tag{3.21}$$

From (3.20)and (3.21) we obtain:

$$\varliminf_{x\to\infty} \frac{S(x)}{\psi(x)} \geq \varlimsup_{\varepsilon\to 0} \varliminf_{x\to\infty} \frac{1}{\psi(x)\varepsilon} \int_x^{x+\varepsilon} [S(x+\varepsilon)-S(y)]dy + A \tag{3.22}$$

and

$$A \geq \varlimsup_{\varepsilon\to 0} \varliminf_{x\to\infty} \frac{1}{\psi(x)\varepsilon} \int_x^{x+\varepsilon} [S(y)-S(x)]dy + \varlimsup_{x\to\infty} \frac{S(x)}{\psi(x)} \tag{3.23}$$

Thus we have the following

THEOREM 89. *Let* $\psi(x) \in \overline{\Psi}_1$ *and let the assumptions on* $S(x), P(x)$ *stand as in theorem 88. Let us assume in addition that*

$$\varlimsup_{\varepsilon\to 0} \varliminf_{x\to\infty} \frac{1}{\psi(x)\varepsilon} \int_x^{x+\varepsilon} (S(x+\varepsilon)-S(y))dy \geq 0 \tag{3.24}$$

and

$$\varlimsup_{\varepsilon\to 0} \varliminf_{x\to\infty} \frac{1}{\psi(x)\varepsilon} \int_x^{x+\varepsilon} (S(y)-S(x))dy \geq 0 \,. \tag{3.25}$$

Then we conclude that

$$\lim_{x\to\infty} \frac{S(x)}{\psi(x)} = A \,.$$

Remarks: If in any of the theorems 87,88,89 we replace the one-sided condition

$$(3.26) \qquad \int_x^{x+1} |dS(y)| - \int_x^{x+1} dS(y) \leq c\,\psi(x)$$

by the two-sided condition

$$(3.27) \qquad \int_x^{x+1} |dS(y)| \leq c\,\psi(x)$$

then $P(x)$ need not be assumed monotone.

In fact the monotoneity of $P(x)$ was only necessary for the positiveness of $g_c(x) = P(x)-P(x-c)$ which in turn was only useful in obtaining (3.27) from (3.26). However, we note that as regards theorem 89, the assumptions (3.24) and (3.25) are presupposed given explicitly, just as in the case of the original condition (3.26).

§4. Averages on $(0,\infty)$

The principal results of the previous section are theorems 83, 84 and 89, which deal with the weighted averages of functions defined on the real line $(-\infty,\infty)$; by a suitable modification, we shall reduce them now to the half-line $(0,\infty)$. We introduce the following

Definition. A function $\phi(t)$ belongs to the class $\overline{\Phi}_p$ if and only if the function $\phi(e^x) \in \overline{\Phi}_p$.

We make the following substitutions in theorems 84 and 83:

$$y = \log\lambda, \quad S(y) = \int^{\lambda} \lambda^{-1} d\, S'(\lambda), \quad g(-y) = \lambda D(\lambda)$$

$$f(-y) = \lambda F(\lambda).$$

Then, theorem 83 runs as follows:

THEOREM 90. Assumptions:

(i) $\phi(t) \in \bar{\Phi}_p$

(ii) $S'(\lambda)$ is of bounded variation in every finite interval $0 < \lambda_0 \leq \lambda_1 < \lambda_2 < \infty$ and

$$\int_t^{et} \frac{|dS'(\lambda)|}{\lambda} \leq c\, \phi(t)$$

(iii) $D(t)$ is a continuous function which is $O\{t^{-1}|\log t|^{-p-3}\}$ both for small and large t, and

$$\frac{1}{\lambda\phi(\lambda)} \int_0^\infty D\left(\frac{\mu}{\lambda}\right) dS'(\mu)$$

exists for every , $0 < \lambda < \infty$ and is bounded,

(iv) $$\int_0^\infty D(t) t^{i\alpha}\, dt \neq 0 \;,\; -\infty < \alpha < \infty \;,$$

(v) there is a constant A such that

$$\lim_{\lambda\to\infty} \{\lambda\phi(\lambda)\}^{-1} \int_0^\infty D\left(\frac{\mu}{\lambda}\right) dS'(\mu) = A \int_0^\infty D(\mu)\, d\mu \;.$$

Conclusion: for any continuous function $F(t)$ which is $O(t^{-1}|\log t|^{-p-2})$ both for small and large t, we have

$$\lim_{\lambda\to\infty} \{\lambda\phi(\lambda)\}^{-1} \int_0^\infty F\left(\frac{\mu}{\lambda}\right) dS'(\mu) = A \int_0^\infty F(\mu)\, d\mu \;.$$

Similarly we have the following

THEOREM 91. Assumptions:

(i) $\phi(t) \in \bar{\Phi}_p$

(ii) $S'(\lambda)$ is real and of bounded variation in every finite interval $0 < \lambda_0 \leq \lambda_1 < \lambda_2 < \infty$ and

$$\int_t^{et} \frac{|dS'(\lambda)|}{\lambda} - \int_t^{et} \frac{dS'(\lambda)}{\lambda} \leq c\,\phi(t)$$

(iii) D(t) _is non-negative, continuous and is_ $O\{t^{-1}|\log t|^{-p-3}\}$ _for both small and large_ t,

(iv) $$\int_0^\infty D(t)t^{i\alpha}\,dt \neq 0, \quad -\infty < \alpha < \infty,$$

(v) _for each_ λ _in_ $0 < \lambda < \infty$ _the integral_

$$\frac{1}{\lambda}\,\phi(\lambda) \int_0^\infty D(\tfrac{\mu}{\lambda})dS'(\mu)$$

exists as a Cauchy limit $\lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}$, _say, and is bounded;_

(vi) _there is a constant_ A _such that_

$$\lim_{\lambda\to\infty} \{\lambda\phi(\lambda)\}^{-1} \int_0^\infty D(\tfrac{\mu}{\lambda})dS'(\mu) = A \int_0^\infty D(\mu)d\mu$$

Conclusion: we have

(vii) $$\int_t^{et} \frac{|dS'(\lambda)|}{\lambda} \leq c\,\phi(t)$$

and hence by theorem 90, we have

(viii) $$\lim_{\lambda\to\infty} \{\lambda\phi(\lambda)\}^{-1} \int_0^\infty F(\tfrac{\mu}{\lambda})dS'(\mu) = A \int_0^\infty F(\mu)d\mu$$

for any continuous function F(t) _which is_ $O(t^{-1}|\log t|^{-p-2})$ _for both small and large_ t.

To reduce theorem 89 to the interval (0,∞) we use slightly different substitutions, namely

$$S(y) = S'(e^y), \qquad P(y) = W(e^{-y}) .$$

We then have the following

THEOREM 92. Assumptions:

(i) $\phi(t) \in \overline{\Phi}_1(t)$

(ii) $S'(\lambda)$ is of bounded variation in every finite interval $0\ \lambda_0 \leq \lambda_1 < \lambda_2 < \infty$ and

(4.1) $$\int_t^{et} |dS'(\lambda)| - \int_t^{et} dS'(\lambda) \leq c\ \phi(t).$$

(iii) the function $W(t)$ is monotone, non-increasing in $0 < t < \infty$, and

$$W(t) = 1 + O(t^a), \qquad t \to 0 ,$$

and

$$W(t) = O(t^{-a}), \qquad t \to \infty$$

for some fixed $a > 0$,

(iv) for $0 < \varepsilon < a$ we can thus form the absolutely convergent integral

$$\chi(\varepsilon + i\alpha) = \int_0^\infty W(t) t^{\varepsilon - 1 + i\alpha}\, dt, \quad -\infty < \alpha < \infty$$

and we assume that

$$\lim_{\varepsilon \downarrow +0} \chi(\varepsilon + i\alpha) \neq 0$$

on $-\infty < \alpha < 0$ and $0 < \alpha < \infty$,

(v) $$\frac{1}{\phi(\lambda)} \int_0^\infty W\left(\frac{\mu}{\lambda}\right) dS'(\mu)$$

exists as a Cauchy limit and is bounded in $0 < \lambda < \infty$,

(vi) there is a constant A such that

(vii) $$\lim_{\lambda\to\infty} \frac{1}{\phi(\lambda)} \int_0^\infty W\left(\frac{\mu}{\lambda}\right)dS'(\mu) = A$$

$$S'(\mu) \to 0 \quad \text{as} \quad \mu \to 0 \quad ;$$

in other words,

$$S'(+0) = S'(0) = 0$$

(viii)

(4.2) $$\overline{\lim_{a\downarrow 1}} \lim_{\lambda\to\infty} \frac{1}{\phi(\lambda)\log a} \int_\lambda^{a\lambda} (S'(a\lambda)-S'(y))\frac{dy}{y} \geq 0,$$

and

(4.3) $$\overline{\lim_{a\downarrow 1}} \lim_{\lambda\to\infty} \frac{1}{\phi(\lambda)\log a} \int_\lambda^{a\lambda} (S'(y)-S'(\lambda))\frac{dy}{y} \geq 0 .$$

Conclusion:

$$\lim_{\lambda\to\infty} \frac{S'(\lambda)}{\phi(\lambda)} = A .$$

§5. *Special cases*

We shall now consider special cases of theorems 90, 91, 92, which are readily recognizable as Tauberian. From theorem 90, we shall obtain the following

THEOREM 93. Let $W(t)$ *be a continuous function defined in* $0 < t < \infty$. *Let* $R(k) > 0$. *Let*

(5.1) $$t^{k-1}W(t) = O(t^{-1}|\log t|^{-3}) \quad \text{for } t\to 0 \text{ and } t\to\infty$$

and

(5.2) $$\int_0^\infty W(t)t^{k-1+i\alpha}\,dt \neq 0 \ , \ -\infty < \alpha < \infty \ .$$

Then if

(5.3) $$\lim_{y\to\infty} y^{-k}\sum_1^\infty a_n\, W\left(\frac{n}{y}\right) = A\int_0^\infty t^{k-1}\, W(t)dt$$

and

(5.4) $$a_n = O(n^{k-1})$$

then

(5.5) $$\lim_{n\to\infty} \frac{k}{n^k} \sum_1^n a_r = A \ .$$

The deduction of theorem 93 from theorem 90 requires two lemmas. The first lemma is a corollary of theorem 93, while the second is purely arithmetical.

LEMMA 9. If

(5.6) $$b_n = O(1)$$

and $G(\lambda)$ is a continuous function satisfying

$$G(\lambda) = O(\lambda^{-1}(\log \lambda)^{-3}), \text{ for } \lambda = 0 \text{ and } \lambda = \infty \ .$$

and

$$\int_0^\infty G(t)t^{i\alpha}\, dt \neq 0, \quad -\infty < \alpha < \infty ,$$

then

$$\lim_{\lambda\to\infty} \{\phi(\lambda)\lambda\}^{-1} \sum_1^\infty b_n\, G(\tfrac{n}{\lambda}) = A \int_0^\infty G(\mu)d\mu$$

implies

(5.7) $$\lim_{n\to\infty} \frac{1}{n\phi(n)} \sum_1^n b_r = A \ .$$

Proof: Set

$$S'(y) = \sum_{r\leq y} b_r, \quad 0 < y < \infty ,$$

then the condition

$$\int_t^{et} \left| \frac{dS'(\lambda)}{\lambda} \right| < c$$

is fulfilled. Also we have

$$\sum_1^\infty b_n\, G(\tfrac{n}{\lambda}) = \int_0^\infty G(\tfrac{\mu}{\lambda})dS'(\mu).$$

By theorem 90, with $\phi(\lambda) \equiv 1$, it follows that

(5.8) $$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{\infty} b_r F(\tfrac{r}{n}) = A \int_0^{\infty} F(\mu)d\mu$$

for every "admissible" function F. Given a fixed $\varepsilon > 0$, we define the function $F_\varepsilon(y) = 1$ for $0 \leq y \leq 1$; $F_\varepsilon(y)$ is continuous and decreases monotonely to zero for $1 \leq y \leq 1+\varepsilon$; and $F_\varepsilon(y) = 0$ for $1+\varepsilon \leq y < \infty$. The function is clearly an "admissible" function. However,

$$\left| \frac{1}{n} \sum_1^{\infty} b_r F_\varepsilon(\tfrac{r}{n}) - \frac{1}{n} \sum_1^{n} b_r \right|$$

$$\leq \frac{c_1}{n} \sum_{n \leq r \leq n+n\varepsilon} F_\varepsilon(\tfrac{r}{n})$$

$$\leq c_1 \int_1^{1+\varepsilon} F_\varepsilon(y)dy$$

$$< c_1 \varepsilon$$

and

$$\left| \int_0^{\infty} F_\varepsilon(\mu)d\mu - 1 \right| = \left| \int_0^{\infty} F_\varepsilon(\mu)d\mu - \int_0^1 F_\varepsilon(\mu)d\mu \right| \leq \varepsilon ;$$

if, therefore, we substitute $F_\varepsilon(y)$ for $F(y)$ in (5.8) and let $\varepsilon \to 0$, we obtain the conclusion (5.7) and the lemma is proved.

<u>LEMMA 10</u>. Let $R(k) > 0$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^{n} \frac{a_r}{r^{k-1}} = A .$$

Then

$$\lim_{n\to\infty} \frac{k}{n^k} \sum_1^{n} a_r = A.$$

Proof: We may suppose A = 0 without loss of generality. If

$$s_n = \frac{1}{n}\sum_1^n \frac{a_r}{r^{k-1}} \quad , \quad \sigma_n = \frac{k}{n^k}\sum_1^n a_r \ ,$$

then

$$\sigma_n = ks_n - \frac{k}{n^k}\sum_{r=1}^n \{r^{k-1}(r-1)-(r-1)^k\} s_{r-1} \ .$$

From this it follows that $s_n \to 0$ implies $\sigma_n \to 0$.

Proof of theorem 93. If we choose

$$b_r = \frac{a_r}{r^{k-1}} \quad \text{and} \quad G(t) = t^{k-1}W(t)$$

then the assumptions of lemma 9 are satisfied and if we now use lemma 10 we obtain the required result. If we put $W(t) = e^{-t}$ we obtain the following

Corollary to theorem 93.

Let $R(k) > 0$. If

$$\lim_{y\to\infty} y^{-k}\sum_1^\infty a_n e^{-n/y} = A.\Gamma(k)$$

and

$$a_n = O(n^{k-1})$$

then

$$\lim_{n\to\infty} \frac{k}{n^k}\sum_1^n a_r = A \ .$$

Remark: Taking $k = 1$ in the above corollary, we obtain: if a sequence $\{a_n\}$ is Abel-summable and $a_n = O(1)$ then it is $(C,1)$ summable.

THEOREM 94. Given a real number $k > 0$, a non-negative non-decreasing function $A(t)$, a function $\phi(t)$ of the class Φ_p, and a non-negative continuous function $W(t)$ for *which*

$$t^{k-1}W(t) = O\{t^{-1}(\log t)^{-p-3}\}$$

for both small and large t, *and*

$$\int_0^\infty W(t)t^{k-1+i\alpha}\,dt \neq 0, \quad -\infty < \alpha < \infty.$$

If

$$\lim_{\lambda\to\infty} \{\phi(\lambda)\lambda\}^{-1} \int_0^\infty W(\tfrac{\mu}{\lambda})dA(\mu) = A \int_0^\infty W(t)t^{k-1}dt \ ,$$

and the left-hand function is bounded in $0 < \lambda < \infty$, *then*

$$k\ A(\lambda) \sim A\ \lambda^k\ \phi(\lambda)$$

as $\lambda\to\infty$.

The proof of theorem 94 depends on theorem 91 together with the following lemma.

LEMMA 11. If $S'(t)$, $\phi(t)$, $D(t)$ and A satisfy all the hypotheses of theorem 91, then

$$S'(\lambda) \sim A\ \lambda\ \phi(\lambda)$$

as $\lambda\to\infty$.

Proof: The assumptions about $F(t)$ of theorem 91 are fulfilled for two non-increasing continuous functions $F_1(t)$, $F_2(t)$ of which the first is 1 in $0 < t \leq 1-\varepsilon$ and 0 for $1 \leq t \leq \infty$, and the second is 1 in $0 < t \leq 1$, and 0 for $1+\varepsilon \leq t < \infty$. The number ε can be chosen so as to make

$$\int_0^\infty \{F_2(t) - F_1(t)\}dt$$

arbitrarily small. The "intermediate" function

$$F_0(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & 1 < t < \infty \end{cases}$$

is not continuous, but it follows from

$$\lim \{\lambda\phi(\lambda)\}^{-1} \int_0^\infty F_1(\tfrac{\mu}{\lambda})dS(\mu) \leq \overline{\lim} \{\phi(\lambda)\lambda\}^{-1} \int_0^\infty F_0(\tfrac{\mu}{\lambda})dS(\mu)$$

$$\leq \lim \{\lambda\phi(\lambda)\}^{-1} \int_0^\infty F_2(\tfrac{\mu}{\lambda})dS(\mu)$$

that the conclusion still holds for it. Substituting F_0 for F in theorem 91, we get the lemma.

Proof of theorem 94. If $\phi(t) \in \overline{\Phi}_p$ then $\phi(t^\delta) \in \overline{\Phi}_p$ for $\delta > 0$. Substituting $A(t)$ for $S(t^k)$, $\phi(t^k)$ for $\phi(t)$, $W(t)$ for $D(t^k)$ and A/k for A, we get theorem 94 from lemma 11.

Corollary to theorem 94.

If $k > 0$ and $W(t)$ is a continuous function in $0 < t < \infty$, $W(t) \geq 0$,

$$t^{k-1}W(t) = O(t^{-1}\log t^{-p-3}), \quad \text{for } t = 0, \text{ and } t = \infty,$$

and

$$\int_0^\infty W(t)t^{k-1+i\alpha}\,dt \neq 0, \quad \text{for } -\infty < \alpha < \infty,$$

and

$$| y^{-k} \sum_1^\infty a_n W(\tfrac{n}{y})| \leq c, \quad 0 < y < \infty,$$

then

$$\lim_{y\to\infty} y^{-k} \sum_1^\infty a_n W(\tfrac{n}{y}) = A\int_0^\infty t^{k-1} W(t)dt$$

and

$$a_n \geq 0$$

imply

$$\lim_{n\to\infty} \frac{k}{n^k} \sum_{1}^{n} a_r = A.$$

Remark: Putting $W(t) = e^{-t}$, we obtain:

if

$$\lim_{y\to\infty} y^{-k} \sum_{1}^{\infty} a_n e^{-n/y} = A.\Gamma(k),$$

and

$$a_n \geq 0$$

then

$$\lim_{n\to\infty} \frac{k}{n^k} \sum_{1}^{n} a_r = A .$$

Taking $k = 1$ we obtain that Abel summability of a sequence of positive terms implies (C,1) summability. (see Chr.I).

As a special case of theorem 92, we have

THEOREM 95. (i) *If $W(t)$ is a monotonic non-increasing function for which*

$$W(t) = \begin{cases} 1 + O(t^a), & \text{at } t = 0 \\ O(t^{-a}), & \text{at } t = \infty \end{cases}$$

for some fixed $a > 0$, and

$$\lim_{\varepsilon\to+0} \int_0^\infty W(t)t^{\varepsilon-1+i\alpha}\, dt \neq 0$$

for $-\infty < \alpha < 0$ *and* $0 < \alpha < \infty$; *and if for a slow-growing function* $\phi(t)$ *of class* $\overline{\Phi}_p$, *we have*

$$\left|\frac{1}{\phi(y)} \sum_{1}^{\infty} a_n W(\tfrac{n}{y})\right| \leq c, \quad 0 < y < \infty ,$$

then the limit-relation

$$\lim_{y\to\infty} \frac{1}{\phi(y)} \sum_{1}^{\infty} a_n W(\tfrac{n}{y}) = A$$

together with the one-sided condition

(5.9) $$a_n \geq -\frac{B}{n}\phi(n)\ , \quad B > 0$$

imply

(5.10) $$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{1}^{n} a_m = A\ ;$$

(ii) in the classical case, we have $\phi(y) \equiv 1$ and then the assumptions

$$\sum_{1}^{\infty} a_n W(\tfrac{n}{y}) \to A, \quad a_n \geq -\frac{B}{n}$$

imply

$$\sum a_n = A\ ;$$

(iii) however, if $\phi(y)\to 0$ as $y\to\infty$, then (5.10) implies first of all

(5.11) $$\sum_{1}^{\infty} a_m = 0$$

and then by combining (5.10) and (5.11) we obtain

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{m=n+1}^{\infty} a_m = -A\ .$$

[Remark on part (iii). In the case $W(t) = e^{-t}$ Tauber's original condition was $a_n = o(\frac{1}{n})$; this is now being rendered more specific by considering $o(1)$ to be a certain $\phi(n)$, and at the same time it is being relaxed into one-sidedness instead of Tauber's original two-sidedness. In

Tauber's original requirement

(5.12) $$\lim_{y\to\infty} \sum_{1}^{\infty} a_n W(\tfrac{n}{y}) = A_0$$

we put $A_0 = 0$, and at the same time we impose a condition on the degree of approximation of the limit towards 0, namely,

(5.13) $$\sum_{1}^{\infty} a_n W(\tfrac{n}{y}) \sim A\, \phi(y) .$$

Now, as a consequence of assumption (5.12) with $A_0 = 0$ we first of all have (5.11) naturally; but we see that our further assumption (5.13) is like-wise productive of a conclusion, namely,

$$r_n \sim -A\, \phi(n)$$

where r_n is the n-th remainder of the series in (5.11)].

Proof of theorem 95. On putting

$$S'(t) = \sum_{n \leq t} a_n$$

we obtain

$$\int_{\lambda}^{e\lambda} |dS'(t)| - \int_{\lambda}^{e\lambda} dS'(t) = \sum_{\lambda < n \leq e\lambda} (|a_n| - a_n),$$

and due to (5.9) this is

(5.14) $$\leq 2B \sum_{\lambda < n \leq e\lambda} \phi(n)/n$$

Now assumption (2.2)" on Φ_1 implies in particular

(5.15) $$g_0\, \phi(\lambda) \leq \phi(n) \leq g_1\, \phi(\lambda), \qquad \lambda < n \leq e\lambda ,$$

and hence (5.14) is

$$\leq 2B\,\phi(\lambda) \sum_{\lambda}^{e\lambda} \frac{1}{n} \leq 4B\,\phi(\lambda)$$

and hence

$$\int_{\lambda}^{e\lambda} |dS'(t)| - \int_{\lambda}^{e\lambda} dS'(t) \leq 4B\,\phi(\lambda)$$

which verifies relation (4.1). But now we have to verify (4.2) and (4.3). Take, for instance, (4.2). By (5.9) we have

$$S(\mu) - S(\lambda) \geq - B \sum_{\lambda \leq n \leq \mu} \frac{\phi(n)}{n}$$

and for $\mu \leq e\lambda$, on account of (5.15), this is

$$\geq - B\,\phi(\lambda) \sum_{\lambda < n \leq \mu} \frac{1}{n} \geq - B\,\phi(\lambda) \log \frac{\mu}{\lambda}$$

therefore

$$\frac{1}{\phi(\lambda)} \int_{\lambda}^{a\lambda} (S(\mu)-S(\lambda))\frac{d\mu}{\mu} \geq -B \int_{\lambda}^{a\lambda} \log \frac{\mu}{\lambda}\,\frac{d\mu}{\mu} \geq - \frac{B}{2}(\log a)^2$$

which obviously gives (4.2) and similarly (4.3).

Remark: Putting $W(t) = e^{-t}$, we obtain:

if

$$\lim_{y \to \infty} \sum_{1}^{\infty} a_n\, e^{-n/y} = A$$

and

$$a_n \geq -B/n$$

then

$$\sum_{1}^{\infty} a_n = A\ .$$

NOTES

Books on Fourier Transforms

S. Bochner, Vorlesungen Über Fouriersche Integrale, Leipzig 1932, Chelsea 1948 (BFI)

T. Carleman, Integrale de Fourier, Uppsala,1944 (CFI)

E.C.Titchmarsh, Introduction to the theory of Fourier integrals, Oxford, 1937(TFI)

N. Wiener, The Fourier Integral and certain of its applications, Cambridge, 1933(WFI)

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Chapter I.

We are noting once for all that many historical and genetic remarks about formulas and concepts concerning Fourier transforms are contained in the "foot-notes" (assembled at the end of BFI).

§2. We will have a peculiar generalization of the Riemann Lebesgue Lemma in Chapter III, §4. It is also possible to prove it differently by the methods of "topological algebra"; however, not too much ought to be made of this abstract alternative method of proof. In the theory of Fourier series there are delicate refinements of the Riemann Lebesgue Lemma to certain types of Fourier Stiletjes Integrals (in the one-dimensional case) and the abstract approach would be hardly applicable to this topic of widening interest.

§3. The convolution-theorem is fundamental to the concept of "random variables" in the theory of probability, and our version of the theorem corresponds to the law of addition for two "independent" random variables, each having a "density".

§4. Laws of "operating" with Fourier transforms of functions in L_1 have been approached systematically in BFI. For L_2, mainly due to Hilbert-space background, they were occurring previously, at least implicitly. They can also be set up for functions in L_p, at any rate, for $1 < p < 2$.

§6. Problems of uniqueness for Fourier series and trigonometric series originated in Riemann's papers, Ges. Werke, 2.Aufl., Leipzig 1892, 227-71,331-44, and were studied rather assiduously by Cantor, Lebesgue, du Bois-Reymond and de la Vallée Poussin. But the corresponding uniqueness theorems for integrals (instead of series) were rather slow in emerging. Our present method of proof has been introduced by G.Polya, Math.Zeit.18(1923)66-108.

A version of uniqueness theorem for functions f(x) not necessarily belonging to L_1 in $(-\infty < x < \infty)$ has been established for the first time by A.C.Offord, Journal. London Math.Soc.,11 (1936)171-4, Proc.London Math.Soc. (2) 42 (1937) 422-80.

§8. Theorem 8 has a universal scheme which is roughly as follows. Given a complete orthogonal system of functions, if we take the corresponding expansion of an arbitrary function, and if the expansion converges reasonably, then the sum-function is the function expanded; but it is very difficult to give a "best possible" version of the theorem; almost any version could be "improved upon" or "generalized".

In theorem 9 the assumption that f(x) belong to L_1 cannot be dispensed with easily, although there ought to be ways of relaxing it to some extent; further research is suggested. The second pre-requisite that f(x) be bounded in a neighborhood of the origin is more rigid, and attempts to soften it would be less promising of success.

§9. Continuity in L_1-norm has been analyzed significantly by Plessner, Journal für Math., 160(1929)26-32. He showed in the one-dimensional periodic case that it characterizes absolutely continuous set-functions among all those of bounded variation. This was then extended in a far reaching manner to multi-dimensional functions by S. Bochner, Annals of Math., 40(1939)769-794, Th.15,789.

§10. Theorem 14 gives a new proof of the uniqueness theorem; so does theorem 7. Although theorem 7 gives a striking method of proving uniqueness, it has the disadvantage (if disadvantage it is) of being peculiar to the

Euclidean set-up, whereas theorem 14 has better chances of extension to non-Euclidean spaces. See S. Bochner and J.Von Neumann, Trans.Amer.Math.Soc., 37(1935)21-50.

§11. Theorem 17 belongs to a type of theorems introduced into Analysis by Esclangon and J.E.Littlewood.

§12. Theorem 18 has been stated mainly for the sake of a later multi-dimensional application, but it has some interest in itself, belonging as it does to the type of theorems proved by Lebesgue, de la Vallée-Poussin and Jackson. See de la Vallée-Poussin's _Leçons sur l'approximation des fonctions d'une variable reelle_, and Jackson's _Theory of approximation_.

§14. Our comparison between Abel (e^{-t}) and Gauss (e^{-t^2}) summability is not novel. See M.L.Cartwright, Proc. London Math.Soc.,(2)31(1930)81-96, but our approach is perhaps more systematic.

§15. See E.Hille, _Functional Analysis and Semi-groups_, p.387.

§16. Except for the admission of slow-growing functions, theorems 25 and 26 are very general versions of Abelian limit-theorems.

§17. Theorem 29 is perhaps new. For a brief explanation of Abelian and Tauberian theorems, see G.H.Hardy, Ramanujan, p.46.

A simple proof of theorem 30 was given by J.Karamata, Math.Zeit.,32 (1930) 319-320.

Chapter II

§2. We are emphasizing that we are proving uniqueness theorem (as in S.Bochner, Math.Annalen,108(1933) 378-410) without securing a general inversion-formula first. In the theory of probability, however, (See H. Cramer's Mathematical methods of Statistics) it is customary to have a general inversion first, from which uniqueness is obtained as a corollary.

§3. Note that in (3.5) the integral is an average over the sphere of radius t with center x, and not an integral over the volume of the sphere; hence the added factor t^{k-1} in (3.6).

§4. Theorem 35 and some subsequent theorems are an introduction to "spherical summability". For a systematic account see S.Bochner, Math.Zeit.34(1931)440-47, Trans. Amer.Math.Soc.40(1936)175-207, Annals of Math.37(1936) 345-56, and K.Chandrasekharan, Bull.Amer.Math.Soc. 52(1946) 474-77, Proc.London Math.Soc.(2)50(1948)210-229, Annals of Math.49(1948)991-1007.

§7. The first proof of theorem 40 was given in BFI, p.187;the second is new.

Chapter III

§4. Theorem 46 and the remarks following it are a counter-part to statements on kernels with du Bois-Reymond singularities. See G.H.Hardy, Quarterly Jour.44(1912/13) 1-40.

§5. The proof of theorem 49, now widely in use, was first given by F.Riesz, Bulletino dell'Unione Matematica Italiana, 1928, p.77.

Chapter IV

§2. For other proofs of Plancherel's theorem, see first of all M.Plancherel, Rend.di Palermo,30(1910)289-335; then there are four proofs listed in TFI,pp.70-83, due respectively to E.C.Titchmarsh, S.Bochner, F.Riesz and N. Wiener; finally, there is a proof in M.H.Stone, Linear Transformations in Hilbert Space, p.104, which is also given in S.Bochner's Princeton Lectures on Fourier Analysis (1936). Our present proof is nearest to the last mentioned with a considerable modification and although it is "operational", it is very much restricted to the Euclidean set-up. Note the general conclusion (3.10) following (3.9); it would be interesting to have similar statements for L_1 and L_p, $1 < p < 2$.

§6. For k-dimensional extensions see I.Halperin, Annals of Math. 38(1937)880-919.

Theorem 60 (and others), including the conditional convergence of (6.8) also work for L_p .

§8. In "operational"calculus this is a special case of a normal operator with a simple spectrum, cf. M.H.Stone. loc.cit.pp.275,311.

Theorem 70 combines Hilbert-space theory with partial ordering, the latter being involved in the statement that an element of Hilbert space in a given realization by point-functions happens to vanish on a describable point - set.

Theorem 73 was given by S.Bochner, Math.Zeit.29(1929) 737-743. Cognate topics were then treated in an extensive paper by T.Kitagawa, Jap.Jour.Math.,13(1937)233-332.

For bounded linear transformations from $L_1(-\infty,\infty)$ into itself, commutative with translations, the general form is

$$T^x f(x) = \int f(x-t)dK(t)$$

where K(t) is of bounded variation in $(-\infty,\infty)$, and this is also the form for bounded continuous functions in $(-\infty,\infty)$ with norm: sup $|f(x)|$.

Theorem 76 is due to N.Wiener, WFI,p.100. Its L_1 analogue is given in WFI,p.97.

Chapter V

§§1,2. Theorem 77 was given by S.Bochner, Annals of Math.35(1934)111-15, with a view to supplying a comprehensive background for the appreciation of the Watson transform. However, it should be noted that while theorem 77(that is, the structural statements on unitary transformations) can be generalized from the half-line $(0,\infty)$ to a general point-set in an Euclidean or more general measure-space, the same is not at all true of the Watson transform. It is a fascinating feature of the Watson theory that the properties peculiar to the ordinary half-line are utilized to the limit, especially the property that the transformation $x = e^y$ links it with the full line $-\infty < y < \infty$, the latter line being both the group-space on which the Fourier(Plancherel) transform is in existence, and the imaginary axis of the complex z-plane on which a Cauchy-Mellin integral of the form (3.25) is available. For the original version of Watson transforms see G.N.Watson, Proc.Lond.Math.Soc.(2)33(1935)156-199.

It would be of some interest to investigate convergence-tests for the validity of (1.13) not as a limit in mean but as a point-limit by ordinary convergence or summability.

The reader should note our insistence on the "conjugate"kernels in (2.2),(2.3) and (2.4), and most important of all, in formula (3.30). For non-real kernels k(x) our transforms are very different from those included in TFI, pp.221-244. In particular, the Hankel transform $k(x) = H_v(x)\sqrt{x}$ with a non-real v is not at all included in our L_2 theorems.

§3. It is somewhat surprising that among all unitary transformations in $L_2(0,\infty)$ the ones described in theorem 82 are the only ones which thus far have been found to have a simple structural property of the kernels k(a,x), l(a,x) corresponding to them. It would be desirable to elucidate the situation by finding some other classes of unitary transformations different from the Watson transform and corresponding to some other simple specializations of the kernels involved.

Chapter VI

§1. A "general" Tauberian theorem is applicable to a general class of kernels. The Tauberian theorem we had in Chr.I is for the special kernel $K(\alpha) = e^{-\alpha}$.

The term "Tauberian theorems" has been used systematically by G.H.Hardy and J.E.Littlewood, even though the main impetus for their study has been furnished by their own work. The general Tauberian theorems of N. Wiener, Annals of Math.33(1932)1-100, changed the character of the subject to a considerable extent by bringing in the theory of Fourier transforms and closure-properties.

The classical theorem of Tauber states that if a series $\sum a_n$ is summable by Abel's method and $a_n = o(\frac{1}{n})$

then $\sum a_n$ converges. See Monats Hefte für Math.8(1897) 273-277. The condition $a_n = o(\frac{1}{n})$ was replaced $a_n = O(\frac{1}{n})$ by J.E.Littlewood, Proc.London Math.Soc.9(1911)434-44; G.H. Hardy and J.E.Littlewood further improved the theorem by having the condition $n\, a_n > -B$ instead of $n\, a_n = O(1)$ Prc. London Math.Soc.(2)13(1914)174-191; E.Landau made the assumption, $S_n = O(1)$ and $\lim_{\delta \to +0} \max_{|m-n| \leq n\delta} |S_m - S_n| = 0$ where $S_n = \sum_1^n a_m$. Rendiconti Palermo,35(1913)265-276. R.Schmidt finally obtained the theorem with the Tauberian condition:

$$\lim_{\delta \to +0} \varliminf_{\substack{n \leq m \leq n(1+\delta) \\ n \to \infty}} (S_m - S_n) \geq 0$$

See Math.Zeit.22(1925)89-152. All these conclusions are implied by theorem 95.

Our treatment of general Tauberian theorems closely follows S.Bochner's in Berliner Sitzungsberichte (1933) 126-144, and in Jour.London Math.Soc.9(1934)141-48, but the general background is supplied by Wiener's paper loc. cit. We have introduced "functions of slow growth" into the averages considered; this has not been done by Wiener; theorem 92 is perhaps new.

It should be noted that in §4 S' does <u>not</u> stand for the derivative of S, but is <u>defined</u> by the given substitution.

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GLOSSARY

Almost periodic function p.49

Composition theorem pp.3,57

Convolution theorem pp.3,5,57

Functions of slow growth p.176

Hankel inversion p.106

Heat-equation p.46

Hilbert space p.104

Inequality, of Hölder and of Minkowski, pp.94,95

Inner product p.104

Isometric transformation p.105

Inversion formula pp.7,10,22,24,65

Karamata extension pp.52,171

Kernels, Abel, Gauss, Fejer, pp.19,34,37

Parseval formula p.67

Riemann Darboux closure p.171

Riemann Lebesgue Lemma pp.3,57,92

Simple type of transformation p.140

Summability, pp.13,25,37,56,61,64,76,98,113

Translation, closure of, pp.142,148,174

Uniqueness pp.11,59

Unitary transformation p.105

Wave-equation p.46

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